CTP TAMU-37/98

UPR/0817-T

October 1998

**hep-th/9810123**

# Spacetimes of Boosted $p$-branes, and CFT in Infinite-momentum Frame

M. Cvetič [†1], H. Lü [†1] and C.N. Pope [‡2]

[†] *Dept. of Physics and Astronomy, University of Pennsylvania, Philadelphia, PA 19104*

[‡] *Center for Theoretical Physics, Texas A&M University, College Station, TX 77843*



ABSTRACT

We study the spacetimes of the near-horizon regions in D3-brane, M2-brane and M5-brane configurations, in cases where there is a pp-wave propagating along a direction in the world-volume. While non-extremal configurations of this kind locally have the same Carter-Novotný-Horský-type metrics as those without the wave, taking the BPS limit results instead in Kaigorodov-type metrics, which are homogeneous, but preserve $\frac{1}{4}$ of the supersymmetry, and have global and local structures that are quite different from the corresponding anti-de Sitter spacetimes associated with solutions where there is no pp-wave. We show that the momentum density of the system is non-vanishing and held fixed under the gravity decoupling limit. In view of the AdS/CFT correspondence, M-theory and type IIB theory in the near-horizon region of these boosted BPS-configurations specifies the corresponding CFT on the boundary in an infinitely-boosted frame with constant momentum density. We model the microstates of such boosted configurations (which account for the microscopic counting of near-extremal black holes in $D = 7$, $D = 9$ and $D = 6$) by those of a boosted dilute massless gas in a $d = 4$, $d = 3$ and $d = 6$ spacetime respectively. Thus we obtain a simple description for the entropy of 2-charge black holes in $D = 7, 9$ and 6 dimensions. The paper includes constructions of generalisations of the Kaigorodov and Carter-Novotný-Horský metrics in arbitrary spacetime dimensions, and an investigation of their properties.

[1] Research supported in part by DOE grant DOE-FG02-95ER40893

[2] Research supported in part by DOE grant DOE-FG03-95ER40917

## Contents

## 1 Introduction

One of the important implications of the non-perturbative aspects of M-theory is the counting of microstates for near-BPS configurations such as black holes and $p$-branes. For near-BPS black holes in $D = 5$ and $D = 4$, this counting can be carried out precisely, both from the D-brane perspective [1, 2, 3, 4, 5] as well as by the counting of the small-scale oscillations of the effective string theory in the NS-NS sector [6, 7, 8, 9, 10]. Interestingly, these examples reduce to the counting of the degrees of freedom of an effective string theory, *i.e.* a (1+1)-dimensional conformal field theory (CFT). (These degrees of freedom are effectively modelled by those of a dilute gas of massless particles in 1+1 dimensions.) On the other hand, the entropy of the D3-brane, M2-brane, and M5-brane, can be modelled (up to a prefactor) by a dilute gas in $d = 4$, $d = 3$ and $d = 6$ respectively [11, 12, 13].

Maldacena's conjecture that relates Type IIB string theory on anti-de Sitter (AdS) spacetime to conformal field theory (CFT) on its boundary (the "AdS/CFT correspondence") [14], which has been investigated by earlier works [11, 15, 16], has initiated broad efforts to test it at the level of the spectrum and correlation functions. One of the important implications of the conjecture is the fresh perspective that it sheds on the microscopics of black holes. It was observed in [17, 18] that the black holes in $D = 5$ and $D = 4$ are related to the three-dimensional BTZ black hole [19] (see also [20]). This leads to a new derivation of black hole entropy[21, 22], by studying the decoupling regime of the near-horizon black hole geometry. The central observation is that, when embedded in a higher-dimensional space, the near-horizon geometry of black holes in $D = 5$ [21, 23] and $D = 4$ [24, 25] contains locally the three-dimensional anti-de Sitter spacetime ($\mathrm{AdS}_3$), whose quantum states are described by a two-dimensional conformal field theory on its asymptotic boundary [26]. The counting of states in this CFT is then used to reproduce the black hole entropy for near-extremal static [21, 24] and rotating [23, 25] black holes in $D = 4$ and $D = 5$ respectively.

In this paper, we address a number of related issues:

- *Near-horizon geometry of boosted $p$-branes/CFT in infinite-momentum frame*

We address the $\mathrm{AdS}_D$/CFT correspondence to cases where there is a pp-wave propagating along a direction in the world-volume of the classical $p$-brane configuration. One has to distinguish two cases, depending upon whether or not the configuration is BPS saturated. In the non-BPS case, the effect of the inclusion of the pp-wave is locally equivalent to performing a Lorentz boost transformation along the direction of propagation of the wave. (If the direction along which the pp-wave propagates is uncompactified, then the equivalence is in fact valid globally, while if the direction is wrapped on a circle, it is only valid locally.) For this reason, $p$-branes with superimposed pp-waves propagating on their world-volumes are often referred to as *boosted* $p$-branes; one should bear in mind though that the global structure may not be precisely describable by a Lorentz boost. In the case of BPS $p$-branes, on the other hand, the inclusion of the pp-wave leads to a metric that is not even locally equivalent to the one where there is no wave. This is because in the BPS limit the Lorentz boost that relates the two metrics becomes singular, corresponding to a boost with velocity approaching the speed of light. Thus in the BPS limit one has two distinct configurations, which are not even locally equivalent, corresponding to the cases with and without the pp-wave. In this case, although the term "boosted $p$-brane" is sometimes used, the expression is somewhat of a misnomer.

For the BPS D3-brane, M2-brane and M5-brane configurations where there is no pp-wave, the near-horizon geometries (corresponding to the decoupling limit) are those of $\mathrm{AdS}_5 \times S^5$, $\mathrm{AdS}_4 \times S^7$ or $\mathrm{AdS}_7 \times S^4$ respectively [27, 28]. Equivalently, one can think of performing a compactification on the 5-spheres, 7-spheres or 4-spheres that foliate the space transverse to the $p$-brane, in which case, in the near-horizon regime, the corresponding AdS spacetimes arise as solutions of the compactified theories. On the other hand, with the inclusion of a pp-wave propagating on the BPS $p$-brane we find that the AdS metric is replaced by a new type of metric, which in four dimensions was first constructed by Kaigorodov [29]. (The four-dimensional metric is of type N in the Petrov classification. See also discussions in [30, 31, 32].) In this paper, we construct arbitrary-dimensional generalisations of the Kaigorodov metric, which include the $D = 5, 4$ and 7 cases arising in the near-horizon regions of the boosted D3-brane, M2-brane and M5-brane. Like AdS, these are homogeneous Einstein metrics, but they differ significantly in both their local and global structures. In particular, although they approach AdS locally at infinity, their boundaries are related to those of the AdS metrics by an infinite Lorentz boost. Thus one may say that the boundary of the generalised Kaigorodov metric is in an infinite-momentum frame. Furthermore, we show that in the gravity decoupling limit, in order to maintain the structure of the Kaigorodov metric, the momentum density (momentum per unit $p$-volume) must be held fixed. (The metric recovers the form of the AdS spactime if instead the momentum density vanishes.) A consequence of this generalisation of the spacetime is that the boundary theory will now be a CFT with an infinite boost, but with a constant momentum density. This new correspondence implies that the entropy of the near-extremal D3-brane or M-brane with pp-wave can be modelled by a dilute massless gas in an appropriately-boosted frame of the world-volume spacetime of the $p$-brane. We show that this is indeed the case, and that the contribution to the entropy in the boosted case is precisely accounted for by the Lorentz contraction factor $1/\gamma$ along the boost direction, implying that the entropy density (entropy per unit $p$-volume) is enlarged by a factor of $\gamma$. (This observation has also been made in [33, 34].)

The situation is somewhat different in the case of non-extremal $p$-branes. We show that the spherical reductions of the configurations with pp-waves give rise to inhomogeneous Einstein metrics, which generalise the Carter-Novotný-Horský metric [35, 36] of four dimensions. We again construct arbitrary-dimensional generalisations, which encompass the cases that arise from the spherical reductions of the D3-brane, M2-brane and M5-brane, and we study some of the pertinent properties of these metrics. As we noted above, in these non-extremal configurations there is locally no distinction between the case where there is a superimposed pp-wave, and the case with no pp-wave. This is because a coordinate transformation allows the harmonic function associated with the pp-wave to be set to unity. Consequently the local form of the Carter-Novotný-Horský metrics is the same whether or not a pp-wave is included in the original $p$-brane solution. The coordinate transformation becomes singular in the extremal limit, which explains why there are two distinct cases in the extremal situation, leading either to the AdS or else to the generalised Kaigorodov metrics after spherical reduction, but only the single case of the generalised Carter-Novotný-Horský metrics in the spherically-reduced non-extremal situations.

We may summarise the situation in the following Table. If we begin with a non-dilatonic $p$-brane in $\widetilde{D}$ dimensions, and perform a dimensional reduction on the foliating $(\widetilde{D}-p-2)$-spheres in the transverse space, then according to whether the $p$-brane is extremal or non-extremal, and whether or not there is a superimposed pp-wave, the lower-dimensional metric (of dimension $n=p+2$) will be of the form:

| | No Wave | Wave |
|---|---|---|
| Extremal | $\mathrm{AdS}_n$ | $\mathrm{K}_n$ |
| Non-extremal | $\mathrm{C}_n$ | $\mathrm{C}_n$ |

Table 1: Spherical reductions of non-dilatonic $p$-branes

Here, $\mathrm{K}_n$ denotes the $n$-dimensional generalisation of the Kaigorodov metric, obtained in Appendix A, and $\mathrm{C}_n$ denotes the $n$-dimensional generalisation of the Carter-Novotný-Horský metric, obtained in Appendix C.

Using the fact that horizon area, and hence entropy, is preserved under dimensional reduction, we show how the entropies of certain of the black holes can be related to the entropies calculated in the associated generalised Kaigorodov or Carter-Novotný-Horský metrics. The results that we obtain in this paper are generalisations of results obtained previously for the BTZ metrics. In particular, the extremal BTZ metric (where the angular momentum $J$ and mass $M$ are related by $J = M\,\ell$, where $-2\ell^{-2}$ is the cosmological constant) is equivalent to $\mathrm{K}_3$, the specialisation of the generalised Kaigorodov metrics to the case $D = 3$. Likewise, the non-extremal BTZ metric is equivalent to $\mathrm{C}_3$, the specialisation of the generalised Carter-Novotný-Horský metrics to the case $D = 3$.

•*Black-hole microstate counting for* $D > 5$.

The above aspect of the AdS/CFT correspondence allows for a study of the microscopics of general static near-extremal black holes in $D = 7$, 9 and 6. In other words, if the entire spatial world-volume of a near-extremal D3-brane, M2-brane or M5-brane configuration with a pp-wave is compactified on a torus, we obtain a two-charge static near-extremal black hole in $D = 7$, $D = 9$ or $D = 6$ respectively. (In $6 \le D \le 9$, such two-charge black holes are generating solutions for the most general black holes of the toroidally compactified heterotic and Type II string theories [37, 38].) We are able to model the statistical entropy of such near-extremal black holes as a boosted dilute gas of massless particles.

Each of the two-form field strengths in a maximal supergravity can be used to construct a single-charge black hole solution. These solutions form a multiplet under the Weyl subgroup of the U-duality group [39]. In $D = 7$, 9 and 6, certain members of the multiplet can be double-dimensionally oxidised to become the D3-brane, M2-brane and M5-brane respectively. The entropy of such a black hole in the near-extremal regime can then be modelled by a dilute gas in the world-volume of the corresponding $p$-brane. In $6 \le D \le 9$, one can construct 2-charge black holes that are generating solutions for the most general black holes. The 2-charge solutions associated with different field configurations also form multiplets under the Weyl subgroup of the U-duality group. Some configurations can be viewed as intersections of $p$-branes in higher dimensions. In this paper, we focus on the cases which correspond to "boosted" D3-brane and M-branes. In other words, the second charge is carried by the Kaluza-Klein vector. In these cases, we show that the contribution to the entropy of the system due to the Kaluza-Klein charge can be understood as a simple consequence of the Lorentz contraction resulting from the boost. Thus the previously-known dilute gas model for the D3-brane and M-branes can be used to understand microscopically the two-charge black holes, except that the dilute gas is now in a boosted frame, rather than in the rest frame. In particular, when the momentum of the system is held fixed as the boost becomes large, it is associated with the Kaluza-Klein charge in the supergravity picture. This observation is consistent with the conjecture that M-theory or type IIB theory on $\mathrm{K}_D \times S^P$ is dual to a CFT in an infinite-momentum frame. It is worth remarking that the approach we have adopted here for studying 2-charge black holes, by considering the case where one of the charges is carried by a Kaluza-Klein vector, seems to be the easiest way of tackling the problem. Other 2-charge black holes, whose higher-dimensional interpretation would be as intersections of $p$-branes, are related to the ones we study here by U-duality transformations. The method we adopt here, combined with U-duality, seems to provide the easiest way for providing an interpretation for the entropy of the black holes that correspond to intersections of $p$-branes,

The paper is organised as follows. In section 2, we discuss the extremal and non-extremal M2-branes, with the inclusion of a pp-wave, and show how their dimensional reductions on $S^7$ give rise to the four-dimensional Kaigorodov and Carter-Novotný-Horský metrics respectively. We compare the entropy and temperature of the $D = 9$ two-charge black hole obtained by compactification on $T^2$ with the corresponding results for the $S^7$ compactification, and show that they agree in the near-extremal regime. In section 3, we generalise the results to the case of the M5-brane and the D3-brane. In section 4, we give a brief discussion of reductions to $D = 3$ and $D = 2$, which include in particular the BTZ black hole in $D = 3$. Appendix A contains our results for the generalisation of the Kaigorodov metric to arbitrary spacetime dimensions, and in Appendix B we construct its Killing vectors and Killing spinors. In Appendix C we generalise the Carter-Novotný-Horský metric to arbitrary dimensions, and construct its Killing vectors.

## 2 M2-brane with a pp-wave

### 2.1 Extremal case

We first consider the intersection of an extremal M2-brane and a gravitational pp-wave in $D = 11$ supergravity. The classical solution is given by

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(-K^{-1}\,dt^2 + K\,(dx_1 + (K^{-1}-1)\,dt)^2 + dx_2^2) + H^{1/3}\,(dr^2 + r^2\,d\Omega_7^2)\ , \\ F_4 &= dt \wedge dx_1 \wedge dx_2 \wedge dH^{-1}\ , \end{aligned} \tag{2.1}$$

$$H = 1 + \frac{Q_1}{r^6}\,, \qquad K = 1 + \frac{Q_2}{r^6}\ .$$

It is worth mentioning that the harmonic function of a single wave in $D = 11$, which would give rise to a D0-brane in $D = 10$, would depend on $(r^2 + x_2^2)^{7/2}$ rather than just $r^6$. The above solution describes a pp-wave, uniformly distributed along the world-volume coordinate $x_2$, and propagating in the direction of the world-volume coordinate $x_1$.

Performing a double-dimensional reduction on the spatial coordinates $x_1$ and $x_2$, one obtains a 2-charge black hole in $D = 9$ maximal supergravity, with the two charges carried by the winding vector $A_{(1)12}$, coming from the dimensional reduction of $A_{(3)}$ in $D = 11$, and the Kaluza-Klein vector $\mathcal{A}^1_{(1)}$. (In this paper, we adopt the notation of [37, 40] for the lower-dimensional fields in maximal supergravities.) Note that in (2.1) we have, for simplicity, chosen the special case where the wave propagates along the $x_1$ direction. In general, the wave can propagate in an arbitrary direction in the $(x_1, x_2)$ plane. This is reflected in the fact that in $D = 9$ maximal supergravity there are two Kaluza-Klein vectors $\mathcal{A}^1_{(1)}$ and $\mathcal{A}^2_{(1)}$, which form a doublet under the $GL(2,\mathbb{R})$ global symmetry of $D = 9$ maximal supergravity. To get the general solution, we can start with the above simple 2-charge solution, involving $\{A_{(1)12}, \mathcal{A}^1_{(1)}\}$, and apply an $SL(2,\mathbb{R})$ global symmetry transformation, under which $A_{(1)12}$ is a singlet. Then we oxidise the solution back to $D = 11$, and thus obtain the solution of the intersection of M2-brane and a wave that propagates on a general direction in the world-volume of the M2-brane. However, since the $GL(2,\mathbb{R})$ global symmetry is nothing but the residual part of the internal general coordinate transformations of $D = 11$ supergravity, it follows that such a wave propagating in a general world-volume direction can be obtained from (2.1) by an appropriate general coordinate transformation. It should be noted however that the general coordinate transformation may have the effect of altering the global structure of the solution.

We are interested in the near-horizon geometry of the M2-brane with pp-wave (2.1). The near horizon is defined to be the regime where $Q_1/r^6 >> 1$, and hence the membrane harmonic function has the form $H \sim Q_1/r^6$ in this region. Note that the size of the non-vanishing wave charge (momentum) $Q_2$ is unimportant, since we have $K \to K - 1$ under the general coordinate transformation [17]

$$t \longrightarrow \tfrac{3}{2}t - \tfrac{1}{2}x_1, \qquad x_1 \longrightarrow \tfrac{1}{2}t + \tfrac{1}{2}x_1\ . \tag{2.2}$$

It is worth mentioning that this near-horizon structure can also be obtained by a number of somewhat different procedures, using U-duality symmetries or T-duality transformations to change the values of the constant terms in the harmonic functions in any $p$-brane solution

[17, 41, 42, 43]. (For the harmonic function $K$, as we have seen, it can be achieved by a mere coordinate transformation.) The simplest way to remove the constant "1" in the harmonic function $H$ in (2.1) is to perform a dimensional reduction of (2.1) on the entire set of three world-volume coordinates of the M2-brane, including the time direction. This gives rise to an instanton solution of an eight-dimensional Euclidean-signatured supergravity, which has an $SL(2,\mathbb{R})$ symmetry that can be used to rescale and shift the harmonic function $H$ by constants while leaving the structure of the solution unaltered [43]. Having performed the symmetry transformation that leads to $H \to H-1$, we can oxidise the solution back to $D=11$, obtaining the near-horizon structure of (2.1). It is not clear however about the significance and the physical interpretation of such a transformation.

The metric of the near horizon of (2.1) is given by

$$ds_{11}^2 = Q_1^{-2/3} r^4 \left(-K^{-1}\,dt^2 + K\,(dx_1 + (K^{-1}-1)dt)^2 + dx^2\right) + Q_2^{1/3} r^{-2} dr^2 + Q_1^{1/3} d\Omega_7^2 \,. \quad (2.3)$$

Thus we see that the spacetime is a product $M_4 \times S^7$. It is of interest to study this new vacuum of M-theory in more detail, and in particular to study the structure of $M_4$. Since the coefficient of the $S^7$ metric $d\Omega_7^2$ is a constant, it follows that $M_4$ must be an Einstein metric, a solution of $D=4$ gravity with a pure cosmological term:

$$e^{-1}\mathcal{L}_4 = R - 2\Lambda \,, \quad (2.4)$$

with $\Lambda = -12 Q_1^{-1/3}$. Here, we choose to take the internal metric to be $ds_7^2 = Q_1^{1/3} d\Omega_7^2$. After the $S^7$ reduction, we obtain the four-dimensional metric

$$ds_4^2 = Q_1^{1/3}\Big(-e^{10\rho}\,dt^2 + e^{-2\rho}\,(dx_1 + e^{6\rho}\,dt)^2 + e^{4\rho}\,dx_2^2 + d\rho^2\Big) \,, \quad (2.5)$$

where $e^\rho = r$. (Note that inside the parentheses we have absorbed the charge parameters by rescaling the world-volume coordinates. See [44] for a detailed discussion of spherical dimensional reduction.)

It is straightforward to verify that (2.5) is an homogeneous Einstein metric, but that it is not AdS$_4$; in fact, it is a metric discovered first by Kaigorodov [29]. We shall denote this metric as K$_4$. In Appendices A and B, we discuss the properties of this metric, and we derive its higher-dimensional generalisations. Included in this is a discussion of the symmetries of the generalised Kaigorodov metrics, and a construction of their Killing spinors and Killing vectors. To be specific, the K$_4$ metric (2.5) has a 5-dimensional isometry group, and it preserves 1/4 of the supersymmetry.

## 2.2 Decoupling limit

It is instructive to study whether there also exists a limit in this M2-brane/wave solution where the field theory on the brane decouples from the bulk. To do this, we note that the metric in (2.1) can be expressed, after the coordinate transformation (2.2), as

$$ds_{11}^2 = H^{-2/3}\,(K\,dx_1^2 + 2dx_1\,dt + dx_2^2) + H^{1/3}\,(dr^2 + r^2 d\Omega_7^2)\ , \tag{2.6}$$

where $H = 1 + Q_1/r^6$ and $K = Q_2/r^6$. The membrane charge $Q_1$ is subject to the Dirac quantisation condition in the presence of a 5-brane. This implies that $Q_1 = N\,\ell_p^6$, where $N$ is an integer and we define the eleven-dimensional Plank length $\ell_p = \kappa_{11}^{2/9}$. The charge $Q_1$ is associated with a momentum density $P$, *viz*, $Q_1 \sim P\,\ell_p^9$. In the asymptotic region $r \to \infty$, the solution (2.6) is Minkowskian, *i.e.* $ds^2 = 2dx_1\,dt + dx_2^2 + dr^2 + r^2 d\Omega_7^2$. Note that in this region $x_1$ and $t$ become light-cone coordinates.

Following [14], we consider the limit $\ell_p \to 0$, while keeping $U = 2r^2/(N\,\ell_p^3)$ fixed. In this limit, we have

$$N\ell_p^6/r^6 >> 1\ , \tag{2.7}$$

and hence we can ignore the constant 1 in $H$. The metric (2.6) becomes

$$ds_{11}^2 = \ell_p^2\,N^{1/3}\,\Big(\tfrac14\Big[\frac{8P}{N^{3/2}}\,\frac{dx_1^2}{U} + U^2\,(2dx_1\,dt + dx_2^2) + \frac{dU^2}{U^2}\Big] + d\Omega_7^2\Big)\ . \tag{2.8}$$

Note that the radius of the Kaigorodov metric is half of that of the seven-sphere. If the momentum density $P$ of the wave vanishes, then $ds^2/\ell_p^2$ is a metric on $\mathrm{AdS}_4 \times S^7$ that depends only on $N$, but is independent of $\ell_p$. The limit where gravity decouples is achieved by taking $\ell_p$ to approach zero [14]. In our case, in order instead to maintain the form of the Kaigorodov metric, the momentum density $P$ must be non-vanishing and fixed. The metric $ds^2/\ell_p^2$ then becomes $\mathrm{K}_4 \times S^7$, which is independent on $\ell_p$.

Thus we see that the decoupling limits in the two cases of the M2-brane and the boosted M2-brane are the same, and in both cases the radius of the seven-sphere is the same, namely $R_7 = N^{1/3}\,\ell_p$. Furthermore, in both cases the momentum density is fixed, but with the difference that in the AdS case the momentum density is zero, whilst in the Kaigorodov case the momentum is non-vanishing. It was conjectured in [14] that M-theory on $\mathrm{AdS}_4 \times S^7$ is dual to a $2+1$ dimensional conformal theory. In the case of $\mathrm{K}_4 \times S^7$, the $\mathrm{K}_4$ can be viewed as infinitely-boosted $\mathrm{AdS}_4$, and the gravitational decoupling limit that maintains the Kaigorodov metric requires that the momentum density remains fixed and non-vanishing. We expect that M-theory on such a metric is dual to the conformal field theory in the infinitely-boosted frame, with constant momentum density. A natural consequence of this conjecture is that the entropy of the boosted M2-brane in the near-extremal regime can be modelled by a dilute gas in a highly-boosted frame for the three-dimensional spacetime, with constant momentum density. We shall show later that this is indeed the case.

### 2.3 Non-extremal case

We now turn our attention to the non-extremal M2-brane with a superimposed gravitational pp-wave. The solution is given by

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(-K^{-1}\, e^{2f}\, dt^2 + K(dx_1 + \coth\mu_2\,(K^{-1}-1)dt)^2 + dx_2^2) \\ &\quad + H^{1/3}(e^{-2f}\, dr^2 + r^2 d\Omega_7^2)\ , \\ A_{(3)} &= \coth\mu_1\, H^{-1}\, dt\wedge dx_1 \wedge dx_2\ , \end{aligned} \tag{2.9}$$

where

$$H = 1 + \frac{\kappa_{11}^{4/3}\, k}{r^6}\,\sinh^2\mu_1\ ,\qquad K = 1 + \frac{\kappa_{11}^{4/3}\, k}{r^6}\,\sinh^2\mu_2\ ,\qquad e^{2f} = 1 - \frac{\kappa_{11}^{4/3}\, k}{r^6}\ , \tag{2.10}$$

The horizon of the boosted M2-brane is at $r_+ = \kappa_{11}^{2/9}\, k^{1/6}$. Note that in this non-extremal case, the effect of the superimposed pp-wave can be removed by a coordinate transformation. Specifically, the coordinate transformation (C.3) given in Appendix C maps the metric (2.9) into the unboosted non-extremal M2-brane, with the metric

$$ds_{11}^2 = H^{-2/3}(-e^{2f}\, dt'^2 + dx_1'^2 + dx_2^2) + H^{1/3}(e^{-2f}\, dr^2 + r^2 d\Omega_7^2)\ . \tag{2.11}$$

Note that the transformation (C.3) is incompatible with any periodic identification of the $x_1$ coordinate. Therefore it is only in the case of a wave propagating along an infinite (*i.e.* unwrapped) world-volume direction on the M2-brane that it can be transformed into a solution with no wave. Note also that the coordinate transformation (C.3), which corresponds to a Lorentz boost in the $(t, x_1)$ plane with velocity $\tanh\mu_2$ (see (C.4)), becomes singular in the extremal limit where $\mu_2 \to \infty$.

The Hawking temperature and entropy per unit 2-volume of the metric (2.9) are easily calculated to be

$$\begin{aligned} T &= \frac{3}{2\pi r_+}\,(\cosh\mu_1\,\cosh\mu_2)^{-1}\ , \\ \frac{S}{L_1\, L_2} &= \frac{\text{Area}}{4\kappa_{11}^2\, L_1\, L_2} \\ &= \frac{k^{7/6}}{4\kappa_{11}^{4/9}}\,\Omega_7\,\cosh\mu_1\,\cosh\mu_2\ . \end{aligned} \tag{2.12}$$

The physical interpretations of the $\mu_1$ and $\mu_2$ dependences in the entropy formula in (2.12) are quite different. When there is no boost, *i.e.* $\mu_2 = 0$, in the near-extremal regime, the entropy can be modelled as dilute gas in the M2-brane world-volume [12, 13], as we shall review presently. On the other hand, the $\mu_2$ dependence is more easily understood. In fact, as we discuss in appendix C, $\cosh\mu_2$ is precisely the $\gamma$-factor of the Lorentz boost (C.3) along the $x_1$ direction, associated with the propagation of the pp-wave (see (C.4)). The entropy of a closed system, which is a measure of the distribution of occupancy numbers of the states, is a Lorentz invariant quantity. However, the entropy density $S/(L_1\, L_2)$ by contrast is not Lorentz invariant, since under the Lorentz boost we have $L_1 \to L_1' = \gamma^{-1}\, L_1$ and $L_2 \to L_2' = L_2$. It follows that under the boost, the new entropy density becomes $S/(L_1'\, L_2')$, which is $\gamma$ times the original density. Thus after a notational change, in which the primed periods $L_i'$ are replaced by the unprimed periods $L_i$, we obtain the entropy formula given in (2.12). This provides a simple explanation for how the entropy depends on the extra charge.[1] In particular, in the near-extremal regime the microscopic entropy of the boosted M2-brane can be modelled by a dilute massless gas in a boosted frame with boost parameter $\gamma = \cosh\mu_2$.

### 2.4 $T^2$ reduction

Since the solution has translational isometries on the world-volume spatial coordinates $(x_1, x_2)$, we can perform dimensional reductions on these two coordinates, thereby obtaining a 2-charge non-extremal isotropic black hole in $D = 9$. The relevant part of the $D = 9$ dimensional Lagrangian that describes this solution is [37, 40]

$$e^{-1}\kappa_9^2\,\mathcal{L} = R - \tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{4}e^{\vec{a}_{12}\cdot\vec{\phi}}\,(F_{(2)12})^2 - \tfrac{1}{4}e^{\vec{b}_1\cdot\vec{\phi}}\,(\mathcal{F}^1_{(2)})^2\,, \tag{2.13}$$

where $\vec{\phi} = (\phi_1, \phi_2)$, $\vec{a}_{12} = (1, 3/\sqrt{7})$ and $\vec{b}_1 = (-3/2, -1/(2\sqrt{7}))$. The $D = 9$ dimensional gravitational constant is related to the one in $D = 11$ as follows

$$\kappa_9^2 = \frac{\kappa_{11}^2}{L_1 L_2}\,, \tag{2.14}$$

[1] Note also that the fact that the charge $Q_2$ has an interpretation as the momentum *density* of the wave in the higher dimension is also easily understood. Upon performing the Lorentz boost, the transformation from the zero-momentum frame to a frame with velocity $v = \tanh\mu_2$ gives a momentum proportional to $\gamma\, v = \cosh\mu_2\, \tanh\mu_2 = \sinh\mu_2$, and the momentum density acquires a further $\cosh\mu_2$ dilatation factor, giving an overall $\sinh 2\mu_2$ dependence. This is exactly the way in which the charge depends on $\mu_2$, as can be seen from (2.16).

where $L_1$ and $L_2$ are the periods of the coordinates $x_1$ and $x_2$ respectively. The non-extremal $D=9$ black hole solution is [45, 46]

$$
\begin{aligned}
& ds_9^2 = -(HK)^{-6/7}\, e^{2f}\, dt^2 + (HK)^{1/7}(e^{-2f}\, dr^2 + r^2 d\Omega_7^2)\ , \\
& \vec{\phi} = \tfrac{1}{2}\vec{a}_{12}\, \log H + \tfrac{1}{2}\vec{b}_1\, \log K\ , \\
& A_{(1)12} = \coth\mu_1\, H^{-1}\, dt, \qquad \mathcal{A}^1_{(1)} = \coth\mu_2\, K^{-1}\, dt\ ,
\end{aligned} \tag{2.15}
$$

which has mass and charges given by

$$
\begin{aligned}
M &= k\,\kappa_{11}^{4/3}(6\sinh^2\mu_1 + 6\sinh^2\mu_2 + 7)L_1\, L_2\,\Omega_7\ , \\
Q_1 = 3k\,\kappa_{11}^{4/3}\,\sinh 2\mu_1\ , &\qquad Q_2 = 3k\,\kappa_{11}^{4/3}\,\sinh 2\mu_2\ ,
\end{aligned} \tag{2.16}
$$

where $\Omega_7$ is the volume of the unit seven-sphere.

The Hawking temperature and entropy are easily calculated to be

$$
\begin{aligned}
T &= \frac{3}{2\pi r_+}(\cosh\mu_1\,\cosh\mu_2)^{-1}\ , \\
S &= \frac{1}{4\kappa_9^2}\, r_+^7\,\Omega_7\,\cosh\mu_1\,\cosh\mu_2 = \frac{k^{7/6}\, L_1\, L_2}{4\kappa_{11}^{4/9}}\,\Omega_7\,\cosh\mu_1\,\cosh\mu_2\ .
\end{aligned} \tag{2.17}
$$

Note that the entropy and temperature are the same as those given in (2.12).

The number of charges of generating solutions of the most general black holes in $D=5$ and $D=4$ are $N=3$ and $N=4$ respectively;[2] their global spacetime structure is the same as that of the Reissner-Nordström black holes, since the moduli of these solutions are finite near the horizon. As a consequence, the entropy is non-vanishing even in the extremal limit. In $D\geq 6$, the number of charges of generating black hole solutions is $N=2$. These black holes are dilatonic, meaning that the dilaton diverges on the horizon in the extremal limit, and consequently the entropy vanishes in the extremal limit. Thus the entropy in the near-extremal regime can best be characterised by its relation to the temperature. The single-charge black holes in $D=9$, $D=7$ and $D=6$ can be viewed as being essentially equivalent to the M2-brane, D3-brane and M5-brane, since they are the double-dimensional reductions of these branes. The entropy and temperature satisfy the ideal-gas relationship $S\sim T^p$ in the near extremal regime, where $p$ is the world-volume spatial dimension of the M-brane or D3-brane [12, 13].[3]

[2] Actually, in $D=4$ the generating solution for the *most* general black holes is specified by $N=5$ charges; however, the metric is still of the Reissner-Nordström-type [6].

[3] The relation is obtained by using the expressions given in (2.17) for the entropy and temperature of the black hole, with $\mu_2$ set to zero (so that there is no boost charge) and taking the limit where $\mu_1$ becomes large, while keeping the charge $Q_1$, given in (2.16), fixed. The approximation becomes a good one when $\cosh\mu_1$ and $\sinh\mu_1$ can be approximated by $\frac{1}{2}e^{\mu_1}$.

When the M2-brane contains also a pp-wave, and hence gives rise to a 2-charge black hole in $D = 9$, the relation changes drastically; it becomes instead $S \sim T^{1/5}$. (The calculation in this case is similar to the previous one, except that now $\mu_2$ is also taken to be large, and both the charges $Q_1$ and $Q_2$ are held fixed.) In general, a 2-charge black hole in maximal supergravity has entropy and temperature that satisfy the relation $S \sim T^{1/(D-4)}$. As we discussed below (2.12), the entropy density formula with the inclusion of the extra Kaluza-Klein charge is precisely accounted for by the effect of the Lorentz contraction along the direction of the boost.

In the next subsection, we shall show that the calculation of the entropy of the 2-charge $D = 9$ black hole in the near-extremal regime can be mapped into a calculation of the entropy of the Carter-Novotný-Horský solution to $D = 4$ gravity with a pure cosmological constant.

## 2.5 $S^7$ reduction

We may now look at the M2-brane with pp-wave from another angle. Note that near to the horizon, we have $H = 1 + (r_+/r)^6 \sinh^2\mu_1$. It follows that we have $H \sim (r_+/r)^6 \sinh^2\mu_1$ in this region, provided that the solution is nearly extremal, namely $\mu_1 >> 1$. In other words, in the near-extremal regime the 1 in harmonic function $H$ can be dropped near the horizon. (As in the case of extremal solutions, the 1 in the harmonic function can in fact be removed by U-duality or T-duality transformations.) As in the extremal case, the metric then becomes a product, $M_4 \times S^7$. Thus we can compactify on the 7-sphere, and obtain a configuration in $D = 4$ that is a solution of Einstein gravity with a pure cosmological term.

It is convenient to take the internal $S^7$ metric to be $ds_7^2 = \kappa_{11}^{4/9}\,(k\sinh^2\mu_1)^{1/3}\,d\Omega_7^2$, *i.e.* the radius of the $S^7$ is

$$R_7 = \kappa_{11}^{2/9}\,(k\sinh^2\mu_1)^{1/6}\ . \tag{2.18}$$

It follows that the four-dimensional gravitational constant is given by

$$\kappa_4^2 = \frac{\kappa_{11}^2}{V_{S^7}} = \frac{\kappa_{11}^{4/9}}{(k\sinh^2\mu_1)^{7/6}\,\Omega_7}\ . \tag{2.19}$$

The four-dimensional metric resulting from the $S^7$ reduction is given by

$$\begin{aligned} ds_4^2 \;&=\; (\kappa_{11}^{4/3}\,k\,\sinh^2\mu_1)^{-2/3}\,r^4\,(-K^{-1}\,e^{2f}\,dt^2 + K(dx_1 + \coth\mu_2\,(K^{-1}-1)\,dt)^2 + dx_2^2) \\ &\quad +(\kappa_{11}^{4/3}\,k\,\sinh^2\mu_1)^{1/3}\,e^{-2f}\,\frac{dr^2}{r^2}\ . \end{aligned} \tag{2.20}$$

This is a solution to the Einstein equations coming from the Lagrangian $e^{-1}\,\kappa^4\,\mathcal{L} = R - 2\Lambda$, where $\Lambda = -12(\kappa_{11}^{4/3}\,k\,\sinh^2\mu_1)^{-1/3}$. The metric (2.20) is no longer homogeneous when the non-extremality factor $e^{2f}$ is present. It is in fact an Einstein metric found by Carter [35], and by Novotný and Horský [36]. (See also [31]).) In the asymptotic regime $r \to \infty$ we have $e^{2f} \to 1$, and the metric becomes the Kaigorodov metric, discussed in the previous section.

The entropy of the solution (2.20) is given by a quarter of the area of the horizon, which is spanned by the spatial coordinates $x_1$ and $x_2$:

$$\begin{aligned} S &= \frac{\text{Area}}{4\kappa_4^2} \\ &= \frac{L_1 L_2}{4\kappa_{11}^{4/9}}\, k^{7/6}\, \Omega_7 \sinh\mu_1 \cosh\mu_2 \,. \end{aligned} \tag{2.21}$$

In the near-extremal limit $\mu_1 >> 1$ we have $\sinh\mu_1 \sim \cosh\mu_1$, and hence the *four*-dimensional entropy (2.21) is the same as the entropy (2.17) of the *nine*-dimensional black hole. Analogously, the Hawking temperature that one calculates for the four-dimensional metric (2.20) is the same as the expression given in (2.17) for the $D = 9$ black hole, except that again the $\cosh\mu_1$ factor is replaced instead by $\sinh\mu_1$. Again, this means that the $D = 9$ black-hole calculation and the $D = 4$ calculation with the Carter-Novotný-Horský metric agree in the near-extremal limit when $\mu_1 >> 1$.

The agreement of the entropies stems from the fact that dimensional reduction and oxidation leave entropies invariant. To see this, let us consider a metric in $\widetilde{D}$ dimensions with an horizon of area $A_{\widetilde{D}}$. The entropy is then given by $S = \widetilde{A}_{\widetilde{D}}/(4\kappa_{\widetilde{D}}^2)$. If we perform a dimensional reduction on an internal space that has volume $V$, to give rise to a $D$-dimensional metric, then the area of the horizon is $A = \widetilde{A}/V$, and hence the entropy becomes $S = A/(4\kappa_D^2) = \widetilde{A}/(4\kappa_D^2\, V)$. Thus the entropy is preserved under dimensional reduction, since $\kappa_{\widetilde{D}}^2 = \kappa_D^2\, V$. It is for this same reason that the entropy of black holes in $D = 5$ and $D = 4$ can be mapped to the problem of BTZ black holes in $D = 3$. Of course the agreement that we are seeing here operates only in the near-extremal regime where $\mu_1 >> 1$, since we made the approximation that the constant term in the harmonic function $H$ could be neglected, in our derivation of the four-dimensional Carter-Novotný-Horský metric by reduction on the seven-sphere.[4]

[4]In [17, 18], the solutions for Reissner-Nordström-type black holes in $D = 5$ and $D = 4$ were mapped into three-dimensional BTZ solutions, where the entropy was shown to agree with the original black-hole results in $D = 5$ and $D = 4$. The mappings were implemented using duality symmetries to shift the constant term in the harmonic function $H$ to zero. Although it was shown in [17, 18] that a mapping could be found that leads to an *exact* agreement for the two entropy calculations, it would seem that other valid mappings could instead have been performed for which the agreement would be seen only in the near-extremal limit.

### 2.6 Microscopic entropy and boosted dilute gas

The correspondence of M-theory on $\mathrm{K}_4 \times S^7$ and CFT on an infinite momentum frame provides a possible microscopic interpretation of 2-charge black holes in $D = 9$. As we have mentioned, in the case where there is no pp-wave propagating on the M2-brane, the entropy and temperature in the near-extremal regime satisfy the ideal-gas relation $S \sim T^2$ in two-dimensional space [12]. When the pp-wave is superimposed in the M2-brane, the entropy is a natural consequence of the Lorentz contraction along the direction of the associated boost, which leads to the appropriate dilation of the entropy density, as discussed in detail in section 2.3. Thus in the near-extremal regime, the entropy can be modelled microscopically as a dilute massless gas in a boosted frame, *viz.* $S = \cosh\mu_2\, S_{\text{dilute gas}}$. This formula applies to any boost. If the boost is finite, then in the limit towards extremality, the momentum of the system becomes zero, and it corresponds to the single-charge solution. If on the other hand, we hold the momentum density $k\, e^{2\mu_2}$ fixed and finite while boosting the system towards speed of light, it corresponds to the near-extremal regime of the 2-charge black hole in $D = 9$, and the extra charge is associated with the momentum. Note that this is a natural consequence of the conjecture that M-theory on the $\mathrm{K}_4 \times S^7$ background is dual to the $2 + 1$ dimensional conformal field theory in an infinitely-boosted frame with constant momentum density, where $\mathrm{K}_4$ is the four-dimensional Kaigorodov metric.

## 3 M5-brane or D3-brane with a pp-wave

### 3.1 M5-brane

The discussion in the previous section can be equally applied to the M5-brane and the D3-brane. We shall first look at the extremal M5-brane in the presence of a pp-wave. The supergravity solution is given by

$$
\begin{aligned}
ds_{11}^2 &= H^{-1/3}(-K^{-1}\, dt^2 + K\,(dx_1 + (K^{-1} - 1)\, dt)^2 + dx_2^2 + \cdots + dx_5^2) \\
&\quad + H^{2/3}\,(dr^2 + r^2\, d\Omega_4^2)\ , \\
F_4 &= *(dH^{-1} \wedge d^6 x), \\
H &= 1 + \frac{Q_1}{r^3}\ , \qquad K = 1 + \frac{Q_2}{r^3}\ ,
\end{aligned}
\tag{3.1}
$$

where $d^6x$ is the volume form on the 5-brane world-volume. To be precise, the solution describes a 5-brane in $D = 11$, with a wave, uniformly distributed on the world-volume coordinates $(x_2, \ldots, x_5)$, and propagating in the world-volume direction $x_1$.

The dimensional reduction of (3.1) on all five spatial 5-brane world-volume coordinates gives rise to a 2-charge black hole in $D = 6$. One charge is the magnetic charge carried by the 4-form field strength $F_4$, and the other is the electric charge carried by the Kaluza-Klein 2-form field strength $\mathcal{F}^1_{(2)}$. We are also interested in the $S^4$ reduction, and, in particular, the reduction in the near-horizon limit $r \to 0$. In this regime, we have $Q_1/r^3 >> 1$, and hence the constant 1 in the harmonic function $H$ can be dropped. The space then becomes a product $\mathrm{K}_7 \times S^4$, *viz*

$$\begin{aligned} ds_{11}^2 &= Q_1^{-1/3} r\,(-K^{-1}\,dt^2 + K\,(dx_1 + (K^{-1}-1)dt)^2 + dx_2^2 + \cdots + dx_5^2) \\ &\quad + Q_1^{2/3} r^{-2} dr^2 + Q_1^{2/3} d\Omega_4^2\,. \end{aligned} \tag{3.2}$$

Compactifying the solution on the $S^4$, with $ds_4^2 = Q_1^{2/3}\,d\Omega_4^2$, we obtain the seven-dimensional Einstein metric

$$ds_7^2 = Q_1^{2/3}\Big( - e^{4\rho}\,dt^2 + e^{-2\rho}\,(dx_1 + e^{3\rho}\,dt)^2 + e^{\rho}\,(dx_2^2 + \cdots dx_5^2) + d\rho^2\Big)\,. \tag{3.3}$$

This is precisely the generalised Kaigorodov metric in $D = 7$, which is derived and its properties discussed in detail in Appendices A and B. The metric (3.3), which we denote by $\mathrm{K}_7$, is homogeneous and Einstein, and is a solution to $D = 7$ gravity with a pure cosmological term[5] $e^{-1}\mathcal{L}_7 = R - 5\Lambda$ with $\Lambda = -24\,Q_1^{-2/3}$.

We shall now consider the limit where the dynamics of the 5-brane decouples from the bulk. Note that we have $Q_1 = N\ell_p^3$ and $Q_2 = P\,\ell_p^9$, where $P$ is the momentum density of the five-dimensional world spatial volume. Following [14], we consider the limit $\ell_p \to 0$ with $U = \frac{1}{2}\sqrt{r/(N\,\ell_p^3)}$ fixed. In this case, we have that $N\ell_p^3/r^3 >> 1$, and hence 1 in function H can be dropped, giving rise to the metric

$$ds_{11}^2 = \ell_p^2\,N^{2/3}\Big(4\Big[\frac{P}{64N^3}\,\frac{dx_1^2}{U^4} + U^2\,(2dx_1\,dt + dx_2^2 + \cdots + dx_5^2) + \frac{dU^2}{U^2}\Big] + d\Omega_4^2\Big)\,. \tag{3.4}$$

Thus the decoupling limit for the M5-brane/wave system is the same as for the pure M5-brane, but giving rise to $\mathrm{K}_7 \times S^4$ or $\mathrm{AdS}_7 \times S^4$ respectively. In the latter case, the momentum density $P$ vanishes, whilst in the former case the momentum density is fixed but non-vanishing. Thus we expect that M-theory on $\mathrm{K}_7 \times S^4$ is dual to the $(0,2)$ conformal theory in an infinitely-boosted frame, with constant momentum density.

[5]Note that in $D$ dimensions, the Einstein-Hilbert action with cosmological term $\mathcal{L} = e\,R - e\,(D-2)\,\Lambda$ gives rise to the Einstein equations $R_{\mu\nu} = \Lambda\,g_{\mu\nu}$. Thus the somewhat unusual-looking normalisation for the cosmological term in the action is needed in order to have a canonical-looking form for the Ricci tensor for the Einstein metric.

The discussion for the non-extremal case is straightforward. The solution in $D = 11$ is given by

$$
\begin{aligned}
ds_{11}^2 &= H^{-1/3}(-K^{-1}\, e^{2f}\, dt^2 + K(dx_1 + \coth\mu_2\,(K^{-1}-1)dt)^2 + dx_2^2 + \cdots + dx_5^2) \\
&\quad + H^{2/3}(e^{-2f} dr^2 + r^2 d\Omega_4^2)\ , \\
F_{(4)} &= \coth\mu_1 * (dH^{-1} \wedge d^6x)\ ,
\end{aligned}
\tag{3.5}
$$

where

$$
H = 1 + \frac{\kappa_{11}^{2/3}\, k}{r^3}\sinh^2\mu_1\ , \qquad K = 1 + \frac{\kappa_{11}^{2/3}\, k}{r^3}\sinh^2\mu_2\ , \qquad e^{2f} = 1 - \frac{\kappa_{11}^{2/3}\, k}{r^3}\ , \tag{3.6}
$$

The horizon of the boosted M5-brane is at $r_+ = \kappa_{11}^{2/9}\, k^{1/3}$. (Again, as we discussed for the M2-brane, locally in this non-extremal case the harmonic function $K$ associated with the wave can be set to 1 by the coordinate transformation (C.3).)

First, let us consider the double-dimensional reduction on the world-volume coordinates $(x_1, x_2, \ldots, x_5)$. This gives rise to a 2-charge non-extremal isotropic black hole in $D = 6$. The relevant Lagrangian is given by [37, 40]

$$
e^{-1}\,\kappa_6^2\,\mathcal{L}_6 = R - \tfrac{1}{2}(\partial\vec{\phi})^2 - \tfrac{1}{48} e^{\vec{a}\cdot\vec{\phi}}\,(F_{(4)})^2 - \tfrac{1}{4} e^{\vec{b}_1\cdot\vec{\phi}}\,(\mathcal{F}^1_{(2)})^2\ , \tag{3.7}
$$

where $\vec{\phi} = (\phi_1, \ldots, \phi_5)$ and

$$
\begin{aligned}
\vec{a} &= (-\tfrac{1}{2}, -3/(2\sqrt{7}), -\sqrt{3/7}, -\sqrt{3/5})\ , \\
\vec{b}_1 &= (-\tfrac{3}{2}, -1/(2\sqrt{7}), -1/\sqrt{21}, -1/\sqrt{15})\ .
\end{aligned}
\tag{3.8}
$$

The six-dimensional gravitational constant is given by $\kappa_6^2 = \kappa_{11}^2/(L_1 \cdots L_5)$, where $L_i$ is the period of the coordinate $x_i$.

The six-dimensional non-extremal black hole is given by [45, 46]

$$
\begin{aligned}
& ds_6^2 = -(H\,K)^{3/4}\, e^{2f}\, dt^2 + (H\,K)^{1/4}(e^{-2f}\, dr^2 + r^2 d\Omega_4^2)\ , \\
& \vec{\phi} - \tfrac{1}{2}\vec{a}\,\log H + \tfrac{1}{2}\vec{b}_1\,\log K\ , \\
& F_{(4)} = \coth\mu_1\, e^{-\vec{a}\cdot\vec{\phi}} * (dH^{-1} \wedge dt) \qquad \mathcal{A}^2_{(1)} = \coth\mu_2\, K^{-1}\, dt\ .
\end{aligned}
\tag{3.9}
$$

It is straightforward to see that its entropy is

$$
\begin{aligned}
S &= \frac{\text{Area}}{4\kappa_6^2}\ , \\
&= \frac{L_1 \cdots L_5}{\kappa_{11}^{10/9}}\, \Omega_4\, k^{4/3}\, \cosh\mu_1\, \cosh\mu_2\ .
\end{aligned}
\tag{3.10}
$$

As in the extremal case, we can also consider the $S^4$ reduction of the non-extremal boosted M5-brane. In particular, we consider the near-extremal case, for which the constant 1 in the harmonic function $H$ can be neglected near the horizon. The space becomes a product $M_7 \times S^4$. The metric of the internal 4-sphere is $ds_4^2 = \kappa_{11}^{4/9}\,(k\sinh^2\mu_1)^{2/3}\,d\Omega_4^2$, and thus its radius is $R_4 = \kappa_{11}^{2/9}\,(k\sinh^2\mu_1)^{1/3}$. It follows that the seven-dimensional gravitational constant is given by

$$\kappa_7^2 = \frac{\kappa_{11}^2}{V_{S^4}} = \frac{\kappa_{11}^{10/9}}{(k\sinh^2\mu_1)^{4/3}\,\Omega_4}\ . \tag{3.11}$$

By performing a dimensional reduction on the four-sphere we arrive at the seven-dimensional metric

$$\begin{aligned} ds_7^2 &= (\kappa_{11}^{2/3}\,k\,\sinh^2\mu_1)^{-1/3}\,r\,\Big(-K^{-1}\,e^{2f}\,dt^2 + K(dx_1 + \coth\mu_2\,(K^{-1}-1)\,dt)^2 \\ &\quad + dx_2^2 + \cdots + dx_5^2\Big) + (\kappa_{11}^{2/3}\,k\,\sinh^2\mu_1)^{2/3}\,e^{-2f}\,\frac{dr^2}{r^2}\ . \end{aligned} \tag{3.12}$$

This solution is still an Einstein metric, but it is no longer homogeneous. It is in fact the seven-dimensional generalisation of the Carter-Novotný-Horský metric, which we obtain in Appendix C. In the asymptotic regime $r \to \infty$ the metric approaches the generalised homogeneous Kaigorodov metric discussed in Appendices A and B. The entropy of the metric (3.12) is given by

$$\begin{aligned} S &= \frac{\text{Area}}{4\kappa_7^2} \\ &= \frac{L_1\cdots L_5}{\kappa_{11}^{10/9}}\,\Omega_4\,k^{4/3}\,\sinh\mu_1\,\cosh\mu_2\ , \end{aligned} \tag{3.13}$$

which agrees with (3.10) in the near-extremal limit $\mu_1 >> 1$. Note that when there is no boost on the M5-brane, the entropy and temperature satisfy the ideal-gas relation $S \sim T^5$ of five-dimensional space, in the near-extremal regime [12]. When the solution is largely-boosted, this relation becomes $S \sim T^{1/2}$. The $\mu_2$ dependence of the entropy density is again the natural consequence of the Lorentz contraction on the world-volume, associated with the boost, and hence the entropy density is enlarged by $\cosh\mu_2$, which is the $\gamma$-factor of the Lorentz boost. Thus the near-extremal entropy can be modelled by a dilute massless gas in a boosted frame. A particular interesting case is to highly boost the dilute gas while hold the momentum density $k\,e^{2\mu_2}$ fixed. This corresponds to the near-extremal 2-charge black holes in $D = 6$. This boosted dilute gas model of the 2-charge black hole entropy is consistent with the conjecture that M-theory on $\mathrm{K}_7 \times S^4$ is dual to the CFT in an infinitely-boosted frame, with constant momentum density.

### 3.2 D3-brane

The D3-brane is supported by the self-dual 5-form in the type IIB theory. The solution for an extremal D3-brane in the presence of a gravitational pp-wave is given by

$$\begin{aligned} ds_{11}^2 &= H^{-1/2}(-K^{-1}\,dt^2 + K\,(dx_1 + (K^{-1}-1)\,dt)^2 + dx_2^2 + dx_3^2) \\ &\quad + H^{1/2}\,(dr^2 + r^2\,d\Omega_5^2)\;, \\ F_5 &= dH^{-1}\wedge d^4x + *(dH^{-1}\wedge d^4x), \\ H &= 1 + \frac{Q_1}{r^4}\;,\qquad K = 1 + \frac{Q_2}{r^4}\;, \end{aligned} \tag{3.14}$$

where $d^4x$ is the volume form on the world-volume of the D3-brane. Note that the wave is uniformly distributed on the plane $(x_2, x_3)$ in the world-volume, and it propagates along the $x_1$ direction.

The dimensional reduction of (3.14) on all three spatial 3-brane world-volume coordinates gives rise to a 2-charge black hole in $D = 7$. In the case instead of the $S^5$ reduction of the near-horizon limit $r \to 0$, the constant 1 in the harmonic function $H$ can be dropped, and the spacetime then becomes a product $\mathrm{K}_5 \times S^5$, where $\mathrm{K}_5$ is the generalised Kaigorodov metric in $D = 5$:

$$\begin{aligned} ds_{10}^2 &= Q_1^{-1/2} r^2\,(-K^{-1}\,dt^2 + K\,(dx_1 + (K^{-1}-1)dt)^2 + dx_2^2 + dx_3^2) \\ &\quad + Q_1^{1/2} r^{-2} dr^2 + Q_1^{1/2} d\Omega_4^2\;. \end{aligned} \tag{3.15}$$

Compactifying the solution on the $S^5$, with $ds_5^2 = Q_1^{1/2}\,d\Omega_5^2$, we obtain the five-dimensional Einstein metric

$$ds_5^2 = Q_1^{1/2}\Big(-e^{6\rho}\,dt^2 + e^{-2\rho}\,(dx_1 + e^{4\rho}\,dt)^2 + e^{2\rho}\,(dx_2^2 + dx_3^2) + d\rho^2\Big)\;. \tag{3.16}$$

This is precisely the generalisation of the Kaigorodov metric to $D = 5$, derived in Appendix A, which is a solution to $D = 5$ gravity with a pure cosmological term $e^{-1}\mathcal{L}_5 = R - 3\Lambda$ with $\Lambda = -16\,Q_1{-}1/2$.

We may again consider the limit where the dynamics of the D3-brane decouples from the bulk. Note that we have $Q_1 = N\ell_p^4$ and $Q_2 = P\ell_p^8$, where $\ell_p = \kappa_{10}^{1/4}$ and $P$ is the momentum density of the world-volume spatial dimensions. Note also that we have $\ell_p^2 = g^{1/2}\,\alpha'$ where $g$ is the string coupling constant. In the limit $\ell_p \to 0$, with $U = r/(\sqrt{N}\,\ell_p^2)$ fixed, one has $N\ell_p^4/r^4 >> 1$, and hence the 1 in the harmonic function $H$ can be dropped [14], giving rise to the metric

$$ds_{10}^2 = \ell_p^2\,N^{1/2}\Big(\frac{P}{N^2}\,\frac{dx_1^2}{U^2} + U^2\,(2dx_1\,dt + dx_2^2 + dx_3^2) + \frac{dU^2}{U^2} + d\Omega_5^2\Big)\;. \tag{3.17}$$

Thus we see that in this limit, the metric $ds_{10}/\ell_p^2$ is independent on $\ell_p$, and the system has a fixed wave-momentum $P$. If $P$ is zero, it reduces to the previously-known $\mathrm{AdS}_5\times S^5$ metric. When $P$ is instead non-vanishing, we have $\mathrm{K}_5\times S^5$. Since the generalised Kaigorodov metric $\mathrm{K}_5$ is an infinitely-boosted $\mathrm{AdS}_5$, we expect that string theory on this $\mathrm{K}_5\times S^5$ background is dual to $N=4$, $D=4$ Yang-Mills theory on an infinitely-boosted frame, with constant momentum density $P$.

Analogously, we may again consider also the non-extremal solution with a superimposed gravitational pp-wave,

$$\begin{aligned} ds_{11}^2 &= H^{-1/2}(-K^{-1}\, e^{2f}\, dt^2 + K(dx_1 + \coth\mu_2\,(K^{-1}-1)dt)^2 + dx_2^2 + dx_3^2) \\ &\quad + H^{1/2}(e^{-2f}dr^2 + r^2 d\Omega_5^2)\ , \\ F_5 &= \coth\mu_1(dH^{-1}\wedge d^4x + {*}(dH^{-1}\wedge d^4x))\ , \end{aligned} \tag{3.18}$$

where

$$H = 1 + \frac{\kappa_{10}\,k}{r^4}\,\sinh^2\mu_1\ ,\qquad K = 1+\frac{\kappa_{10}\,k}{r^4}\,\sinh^2\mu_2\ ,\qquad e^{2f} = 1-\frac{\kappa_{10}\,k}{r^4}\ . \tag{3.19}$$

The horizon of the non-extremal boosted D3-brane is at $r_+ = \kappa_{10}^{1/4}\,k^{1/4}$. As for the non-extremal M-branes discussed previously, the coordinate transformation (C.3) locally maps the solution (3.18) to the unboosted one, where $K=1$.

First, let us consider the double-dimensional reduction on the world-volume coordinates $(x_1,x_2,x_3)$. This gives rise to a 2-charge non-extremal isotropic black hole in $D=7$. The relevant Lagrangian that describes this solution is

$$e^{-1}\,\kappa_7^2\,\mathcal{L}_7 = R - \tfrac12(\partial\vec\phi)^2 - \tfrac{1}{44}e^{\vec a_{34}\cdot\vec\phi}\,(F_{(2)34})^2 - \tfrac14 e^{\vec a_{12}\cdot\vec\phi}\,(F_{(2)12})^2\ , \tag{3.20}$$

where $\vec\phi = (\phi_1,\ldots,\phi_4)$ and

$$\begin{aligned} \vec a_{12} &= (1, 3/\sqrt7, -1/\sqrt{21}, -1/\sqrt{15})\ , \\ \vec a_{34} &= (-\tfrac12, -3/(2\sqrt7), 4/\sqrt{21}, 4/\sqrt{15})\ . \end{aligned} \tag{3.21}$$

The seven-dimensional gravitational constant is given by $\kappa_7^2 = \kappa_{10}^2/(L_1L_2L_3)$, where $L_i$ is the period of the coordinate $x_i$.

The seven-dimensional non-extremal 2-charge black hole is given by

$$\begin{aligned} & ds_7^2 = -(H\,K)^{4/5}\,e^{2f}\,dt^2 + (H\,K)^{1/5}(e^{-2f}\,dr^2 + r^2 d\Omega_5^2)\ , \\ & \vec\phi = \tfrac12\vec a_{34}\,\log H + \tfrac12\vec a_{12}\,\log K\ , \\ & A_{(1)34} = \coth\mu_1\,H^{-1}\,dt\ ,\qquad A_{(1)12} = \coth\mu_2\,K^{-1}\,dt\ . \end{aligned} \tag{3.22}$$

It is straightforward to see that the entropy of the black hole is

$$\begin{aligned} S &= \frac{\text{Area}}{4\kappa_7^2}\,, \\ &= \frac{L_1 L_2 L_3}{\kappa_{10}^{3/4}}\,\Omega_5\,k^{5/4}\,\cosh\mu_1\,\cosh\mu_2\,. \end{aligned} \tag{3.23}$$

Let us now consider instead the $S^5$ reduction of the boosted D3-brane. In the near-extremal limit, the constant 1 in $H$ can be dropped near the horizon, and hence the spacetime becomes a product $M_5 \times S^5$. The metric of the internal 5-sphere is $ds_5^2 = \kappa_{10}^{1/2}\,(k\sinh^2\mu_1)^{1/2}\,d\Omega_5^2$, and so its radius is $R_5 = \kappa_{10}^{1/4}\,(k\sinh^2\mu_1)^{1/4}$. It follows that the five-dimensional gravitational constant is given by

$$\kappa_5^2 = \frac{\kappa_{10}^2}{V_{S^5}} = \frac{\kappa_{11}^{3/4}}{(k\sinh^2\mu_1)^{5/4}\,\Omega_5}\,. \tag{3.24}$$

Implementing the $S^5$ reduction, we obtain the five-dimensional generalisation of the Carter-Novotný-Horský metric, derived in Appendix C:

$$\begin{aligned} ds_5^2 &= (\kappa_{10}\,k\,\sinh^2\mu_1)^{-1/2}\,r^2\Big(-K^{-1}\,e^{2f}\,dt^2 + K(dx_1 + \coth\mu_2\,(K^{-1}-1)\,dt)^2 \\ &\quad + dx_2^2 + dx_3^2\Big) + (\kappa_{10}\,k\,\sinh^2\mu_1)^{2/3}\,e^{-2f}\,\frac{dr^2}{r^2}\,. \end{aligned} \tag{3.25}$$

Again this solution is still Einstein, but it is no longer homogeneous. In the asymptotic region $r \to \infty$, the metric approaches the generalised homogeneous Kaigorodov metric. The entropy of the metric (3.25) is given by

$$\begin{aligned} S &= \frac{\text{Area}}{4\kappa_5^2} \\ &= \frac{L_1 L_2 L_3}{\kappa_{10}^{3/4}}\,\Omega_4\,k^{5/4}\,\sinh\mu_1\,\cosh\mu_2\,, \end{aligned} \tag{3.26}$$

which agrees with (3.23) in the near-extremal limit $\mu_1 >> 1$. In the case where the wave is absent, the entropy and temperature satisfy the ideal-gas relation $S \sim T^3$; however, the presence of the wave alters the relation, and it becomes $S \sim T^{1/3}$. As in the previous cases, the $\mu_2$ dependence of the entropy density is a natural consequence of the Lorentz contraction along the direction of the boost on the world-volume, implying that the entropy density is enlarged by the factor $\cosh\mu_2$, which is the $\gamma$-factor of the Lorentz boost. Thus in the near extremal regime, the system can be modelled by an ideal gas in a boosted frame in four dimensional spacetime. When the system is highly boosted, but with the momentum density $k\,e^{2\mu_2}$ held fixed, then it gives rise to the entropy of the near-extremal 2-charge black hole in $D = 7$. This is consequence of the correspondence that type IIB supergravity on $K_5 \times S^5$ is dual to the $D = 4$, $N = 4$ Yang-Mills theory in an infinitely-boosted frame, with constant momentum density, where $K_5$ is the five-dimensional generalisation of the Kaigorodov metric.

## 4 Dyonic string with pp-wave

Lower dimensional examples such as $AdS_3$ and $AdS_2$ also arise as the near-horizon limits of supergravity $p$-branes. $AdS_3 \times S^3$ is the near horizon of the dyonic string in $D = 6$. When a wave is propagating on the worldsheet of the string with momentum density $P$, it gives rise to a 3-charge non-dilatonic black hole in $D = 5$. The metric of the spacetime $AdS_3 \times S^3$ plus a wave is given by

$$ds_6 = \ell_p^2 \sqrt{N_1 N_2}\Big(\frac{P}{N_1 N_2}\, dx^2 + \frac{2r^2}{N_1 N_2\, \ell_p^4}\, dx\, dt + \frac{dr^2}{r^2} + d\Omega_3^2\Big)\ , \tag{4.1}$$

where $\ell_p = \kappa_6^{1/2}$, and $N_1$, $N_2$ are the electric and magnetic charges of the dyonic string, and $P$ is the momentum density of the wave. Note that if $P = 0$, the three-dimensional metric obtained by dimensional reduction on $S^3$ is $AdS_3$, in horospherical coordinates. When $P$ is non-vanishing, owing to the degeneracy of three-dimensional gravity,[6] the metric is still locally $AdS_3$, but the global structure is different; this is the $D = 3$ case of the generalised Kaigorodov metrics obtained in Appendix A. This metric is equivalent to the BTZ black-hole metric [19], in the extremal limit where $J = M\,\ell$ (here $-2\ell^{-2}$ is the cosmological constant). The boundaries of the $AdS_3$ and three-dimensional Kaigorodov (or extremal BTZ) metrics are different: In horospherical coordinates, the boundary of $AdS_3$ is a two-dimensional Minkowski spacetime, whilst in the above metric, the boundary is two-dimensional a spacetime in the infinite-momentum frame. The gravitational decoupling limit is $\ell_p \to 0$, while keeping $U = r/\sqrt{N_1\, N_2\, \ell_p^4}$ fixed.

The different global structures of the horospherical $AdS_3$ and the spacetime arising in the case where there is a pp-wave can also be seen from the entropy/temperature relation in the near-extremal regime. When there is a wave propagating on $AdS_3$, we have $S \sim T^0$. On the other hand when the wave is absent, we have $S \sim T$, which is the ideal-gas relation in one dimension. When the extra Kaluza-Klein charge is included, the new parameter $\mu_2$ in (3.26) is again the natural consequence of the associated Lorentz contraction of the volume, implying the dilation of the entropy density by a factor of $\gamma = \cosh \mu_2$. Thus the

[6] In three dimensions the Riemann tensor is characterised completely by the Ricci tensor, and consequently any three-dimensional Einstein metric with negative cosmological constant is locally equivalent to $AdS_3$.

microscopic entropy of the classical solution with the presence of the wave can be modelled by a dilute massless gas in a two-dimensional spacetime in a boosted frame. In particular the three-charge BPS black hole in $D = 5$ has non-vanishing entropy, which corresponds microscopically to a dilute gas in an infinitely-boosted frame, but with the momentum density held fixed.

$\mathrm{AdS}_2$ spacetime arises in supergravity as the near-horizon geometry of the extremal Reissner-Nordström-type black hole in $D = 5$ and $D = 4$. Its boundary is one dimensional, and hence there can be no propagating wave.

# 5 Conclusions and Discussion

In this paper, we have studied the three cases of single-charge non-dilatonic $p$-branes, namely the M2-brane and M5-brane of M-theory, and the D3-brane of the type IIB string, in the presence of a gravitational pp-wave propagating in the world-volume. When dimensionally reduced on all the spatial world-volume coordinates, these configurations give rise to 2-charge black holes in $D = 9$, 6 and 7. One of the charges comes from the original 4-form or 5-form antissymmetric tensor charge in $D = 11$ or $D = 10$, while the other is carried by a Kaluza-Klein vector. If the configuration is non-extremal, the effect of the inclusion of the pp-wave is locally equivalent to a Lorentz boost on the world-volume of the $p$-brane, but for extremal configurations the corresponding boost would be singular, with a boost velocity equal to the speed of light.

The near-horizon structure of the M2-brane, M5-brane or D3-brane with a pp-wave is of a product form, $M_4 \times S^7$, $M_7 \times S^4$ or $M_5 \times S^5$, where in the extremal case $M_n$ is the $n$-dimensional generalisation $\mathrm{K}_n$ of the four-dimensional Kaigorodov metric. In the non-extremal case $M_n$ is the $n$-dimensional generalisation $\mathrm{C}_n$ of the four-dimensional Carter-Novotný-Horský metric. The metrics $\mathrm{K}_n$, which we construct in Appendix A, are homogeneous Einstein metrics. The metrics $\mathrm{C}_n$, which we construct in Appendix C, are inhomogeneous Einstein metrics. Since the local structure of the non-extremal $p$-branes is the same whether or not there is a pp-wave present, there are only global differences between the structures of the generalised Carter-Novotný-Horský metrics that correspond to the $p$-branes with and without the pp-wave. On the other hand in the extremal case there is no non-singular boost that can relate the solution with the pp-wave to the one without, and for this reason the generalised Kaigorodov metric $\mathrm{K}_n$ is not even locally the same as the $\mathrm{AdS}_n$ metric which would arise in the $M_n\times$Sphere product in the near-horizon limit of the extremal $p$-brane with no pp-wave. In Appendix B we construct the Killing vectors and Killing spinors on the generalised Kaigorodov metrics. In particular, we find that $\mathrm{K}_n$ admits just 1/4 of the maximal number of Killing spinors that occur on $\mathrm{AdS}_n$.

We have argued that by considering the extremal M2-brane, M5-brane or D3-brane in the presence of a pp-wave, the conjectured relations between supergravity in $\mathrm{AdS}_n$ and conformal field theories on its boundary can be generalised to relations between supergravity on the Kaigorodov-type metric $\mathrm{K}_n$ and a CFT on its boundary. Specifically, this boundary is related to the usual $\mathrm{AdS}_n$ boundary by an infinite Lorentz boost, and so the expected conformal field theories will now be related to those of the $\mathrm{AdS}_n$ backgrounds by a singular passage to the infinite-momentum frame. The decoupling limit, where the gravitational constant is sent to zero, requires holding the momentum density of the wave fixed. This correspondence is consistent with the supersymmetries of the two theories. In the supergravity picture, the Kaigorodov metric preserves just 1/4 of the supersymmetry. On the rest-frame CFT side, the superconformal invariance enhances supersymmetry by doubling the number of conserved supercharges. In the infinitely-boosted frame, the non-vanishing momentum implies that half of the original supersymmetry, as well as the superconformal symmetry is also broken. Thus it follows that the theory has just 1/4 of the conserved supercharges.

We also considered the macroscopic and microscopic entropies of 2-charge black holes in $D = 9$, 7 and 6 in their near-extremal regimes. We showed that these entropies are related to those of the corresponding single-charge black holes by factors that can be accounted for as Lorentz contractions of the world-volume along the direction of the boost that relates the solutions with and without the pp-wave. Consequently, the microscopic entropy of such a near-extremal black hole can be described in terms of a boosted dilute gas of massless particles on the world-volume of the original $p$-brane. In other words, we have $S = \cosh\mu_2\, S_{\text{dilute gas}}$, for any boost parameter $\gamma = \cosh\mu_2$. (We also showed that the macroscopic entropy of the 2-charge black holes in their near-extremal regimes can also be calculated in the associated generalised Carter-Novotný-Horský metrics, obtained by dimensionally-reducing the original $p$-brane plus pp-wave solutions on the foliating spheres of the transverse space.) If we consider a dilute gas on the world-volume of the $p$-brane in a highly-boosted frame, but with the momentum density fixed, then the entropy and temperature satisfy a relation $S \sim T^{1/(\tilde{d}-2)}$, where $\tilde{d}$ is the dimension of the foliating sphere of the space transverse to the $p$-brane. This observation suggests that the co-dimension $(\tilde{d}+1)$ of the $p$-brane seems to be encoded in the CFT theory on an infinitely-boosted frame with constant momentum density.

It is worth remarking that the Kaigorodov metric in $D = 4$ arises as the near-horizon geometry of the intersection of an M2-brane and a pp-wave, which is the oxidisation of a ten-dimensional D0-brane. This may lead to a connection between the CFT and the M(atrix) model on an AdS background.

We should like to conclude with proposals for future study, beyond the scope of this paper. In the CFT in the infinitely-boosted frame, the only surviving states are those with purely transverse polarisations, and their correlation functions should reflect this fact. Note also that in the rest-frame CFT the correlation functions are usually calculated in a Euclideanised spacetime, while in the current context the Minkowskian nature of the spacetime becomes crucial. On the dual side, this information about the field excitations is encoded in the perturbations of the background of the Kaigorodov-type metrics (the near-horizon region of BPS $p$-branes with pp-waves). It is thus of interest to address these issues both in the CFT and on the gravity side, in order to shed further light on the nature of the correspondence in the infinitely-boosted frame.

## Acknowledgments

We are very grateful to Gary Gibbons and Steven Siklos for discussions about the Kaigorodov metric, to Igor Klebanov and Juan Maldacena for discussions on CFT in boosted frames, and decoupling limits, and Glen Agnolet, Mike Duff, Zachary Guralnik, Randy Kamien, Tom Lubensky and Akardy Tseytlin for discussions.

## Appendices

## A $D$-dimensional generalisation of the Kaigorodov metric

Let us consider the following family of metrics in $D = n + 3$ dimensions:

$$ds^2 = -e^{2a\rho}\, dt^2 + e^{2b\rho}\, (dx + e^{(a-b)\rho}\, dt)^2 + e^{2c\rho}\, dy^i\, dy^i + d\rho^2 \ , \tag{A.1}$$

where $a$, $b$ and $c$ are arbitrary constants. It is easily seen that these encompass the metrics that we obtained in this paper by the spherical dimensional reduction of the extremal M2-brane, M-5-brane and D3-brane with pp-waves. Choosing the natural orthonormal basis

$$e^0 = e^{a\rho}\, dt \ , \qquad e^1 = e^{b\rho}\, (dx + e^{(a-b)\rho}\, dt) \ , \qquad e^2 = d\rho \ , \qquad e^i = e^{c\rho}\, dy^i \ , \tag{A.2}$$

where $3 \le i \le n{+}2$, we find that the torsion-free spin connection, defined by $de^a = -\omega^a{}_b \wedge e^b$, $\omega_{ab} = -\omega_{ba}$, is given by

$$\begin{aligned}
&\omega_{01} = \tfrac{1}{2}(a-b)\, e^2 \ , \qquad \omega_{0i} = 0 \ , \qquad \omega_{02} = -a\, e^0 + \tfrac{1}{2}(a-b)\, e^1 \ , \\
&\omega_{12} = b\, e^1 + \tfrac{1}{2}(a-b)\, e^0 \ , \qquad \omega_{1i} = 0 \ , \qquad \omega_{2i} = -c\, e^i \ , \qquad \omega_{ij} = 0 \ .
\end{aligned} \tag{A.3}$$

It is immediately evident from this that the metrics are homogeneous, since all orthonormal components of the spin connection, and hence of the curvature, are constants. Consequently, all curvature invariants are constants. We find that the curvature 2-forms, defined by $\Theta_{ab} = d\omega_{ab} + \omega_a{}^c \wedge \omega_{cb}$, are given by

$$\begin{aligned}
&\Theta_{01} = \tfrac{1}{4}(a+b)^2\, e^0 \wedge e^1 \ , \qquad \Theta_{02} = \tfrac{1}{4}(a^2 + 6ab - 3b^2)\, e^0 \wedge e^2 - b(a-b)\, e^1 \wedge e^2 \ , \\
&\Theta_{0i} = ac\, e^0 \wedge e^i - \tfrac{1}{2}c(a-b)\, e^1 \wedge e^i \ , \qquad \Theta_{ij} = -c^2\, e^i \wedge e^j \ , \\
&\Theta_{12} = -\tfrac{1}{4}(a^2 - 2ab + 5b^2)\, e^1 \wedge e^2 - b(a-b)\, e^0 \wedge e^2 \ , \\
&\Theta_{1i} = -bc\, e^1 \wedge e^i - \tfrac{1}{2}c(a-b)\, e^0 \wedge e^i \ , \qquad \Theta_{2i} = -c^2\, e^2 \wedge e^i \ .
\end{aligned} \tag{A.4}$$

From this, we find that the Ricci tensor has the vielbein components

$$\begin{aligned}
&R_{00} = \tfrac{1}{2}a^2 + 2ab - \tfrac{1}{2}b^2 + n\, ac \ , \qquad R_{11} = -\tfrac{1}{2}a^2 - \tfrac{3}{2}b^2 - n\, bc \ , \\
&R_{22} = -\tfrac{1}{2}(a+b)^2 - n\, c^2 \ , \qquad R_{ij} = -(a + b + n\, c)\, c\, \delta_{ij} \ , \\
&R_{01} = -\tfrac{1}{2}(a-b)\,(2\, b + n\, c) \ .
\end{aligned} \tag{A.5}$$

Requiring that the metrics be Einstein, namely that the vielbein components of the Ricci tensor obey $R_{ab} = \Lambda\, \eta_{ab}$, we find that there are exactly two inequivalent solutions, *viz.*

$$\mathrm{AdS}_{n+3}: \qquad a = b = c = 2L \ , \tag{A.6}$$

$$\mathrm{K}_{n+3}: \qquad a = (n+4)\, L \ , \qquad b = -n\, L \ , \qquad c = 2\, L \ , \tag{A.7}$$

where $L = \frac{1}{2}\sqrt{-\Lambda/(n+2)}$ and the cosmological constant $\Lambda$ is negative. The first family of Einstein metrics corresponds to anti-de Sitter spacetime in $D = n+3$, while the second family corresponds to $D = n+3$ homogeneous Einstein metrics that generalise the Kaigorodov metric of four dimensions. Note that this second family, of generalised Kaigorodov metrics, can be written in the form

$$ds^2 = e^{-2nL\rho}\, dx^2 + e^{4L\rho}\,(2dx\, dt + dy^i\, dy^i) + d\rho^2 \ . \tag{A.8}$$

Substituting the constants $a$, $b$ and $c$ given by (A.7) into (A.4), we find that the curvature 2-forms $\Theta_{ab}$ for the generalised Kaigorodov metrics can be written in terms of the Weyl

2-forms $C_{ab} = \frac{1}{2} C_{abcd}\, e^c \wedge e^d$, where $C_{abcd}$ is the Weyl tensor, as follows

$$\Theta_{ab} = -4L^2\, \eta_{ac}\, \eta_{bd}\, e^c \wedge e^d + C_{ab}\ . \tag{A.9}$$

Here, the Weyl 2-forms are given by

$$\begin{aligned}
&C_{12} = -C_{02} = n\, \mu\, (e^0 - e^1) \wedge e^2\ ,\\
&C_{0i} = -C_{1i} = \mu\, (e^0 - e^1) \wedge e^i\ ,\\
&C_{01} = C_{ij} = C_{2i} = 0\ ,
\end{aligned} \tag{A.10}$$

where $\mu = 2(n+2)\, L^2$. Thus we see that the Weyl tensor is non-zero in the generalised Kaigorodov metrics, although it has a a rather simple structure. Note that the vielbein combination $e^0 - e^1$ that appears in all the non-vanishing Weyl 2-form components is simply given by $e^0 - e^1 = -e^{b\rho}\, dx$. The vector dual to the 1-form $-(e^0 - e^1)$ is simply $K_{(0)} = \partial/\partial t$. This is a null Killing vector, and a zero eigenvector of the Weyl tensor, satisfying $C_{abcd}\, K^d_{(0)} = 0$. The generalisations of the Kaigorodov metric that we have obtained here can be interpreted as describing gravitational waves propagating in an anti-de Sitter spacetime background. (This is discussed for four-dimensional Kaigorodov metric itself in [32].)

In the case of four dimensions, the Kaigorodov metric is of type N in the Petrov classification (see, for example, [31]). If we define the dual of the Weyl tensor by $\widetilde{C}_{abcd} = \frac{1}{2} \epsilon_{abef}\, C^{ef}{}_{cd}$, and thence the complex Weyl tensor $W_{abcd} \equiv C_{abcd} + \mathrm{i}\, \widetilde{C}_{abcd}$, then it is easily seen that we can write $W_{abcd}$ in the null form $W_{abcd} = -4 V_{ab}\, V_{cd}$, where the 2-form $V = \frac{1}{2} V_{ab}\, e^a \wedge e^b$ is given by

$$V = \sqrt{\frac{-\Lambda}{8}}\, (e^0 - e^1) \wedge (e^2 - \mathrm{i}\, e^3)\ . \tag{A.11}$$

The null Killing vector $K_{(0)} = \partial/\partial t$ is the quadruple Debever-Penrose null vector of the type-N Weyl tensor [32].

In three dimensions, the Kaigorodov metric becomes simply $ds^2 = dx^2 + 2e^{4L\rho}\, dx\, dt + d\rho^2$. This can be seen to be equivalent to the extremal limit of the BTZ black-hole metric described in [19], where the angular momentum $J$ and mass $M$ are related by $J = M\, \ell$, and $-2\ell^{-2}$ is the cosmological constant.

The family of Einstein metrics in (A.6), by contrast, corresponds to the AdS metrics, with

$$\begin{aligned}
ds^2 &= e^{4L\rho}\, (-dt^2 + (dx + dt)^2 + dy^i\, dy^i) + d\rho^2\ ,\\
&= e^{4L\rho}\, (-dt^2 + dx'^2 + dy^i\, dy^i) + d\rho^2\ ,
\end{aligned} \tag{A.12}$$

where in the second line we have made the coordinate redefinition $x' = x + t$ to give the metric its standard horospherical form. The Weyl tensor of course vanishes for this solution, and so then the curvature 2-forms are simply given by (A.9) with $C_{ab} = 0$.

Note that the AdS metrics can be obtained from the generalised Kaigorodov metrics by taking an appropriate singular limit. If we make the redefinitions

$$x \longrightarrow \tfrac{\lambda}{\sqrt{2}}\,(x+t)\ , \qquad t \longrightarrow \tfrac{1}{\lambda\sqrt{2}}\,(x-t)\ , \tag{A.13}$$

to the coordinates $x$ and $t$ appearing in the generalised Kaigorodov metrics (A.8), and then send the constant $\lambda$ to zero, we find that (A.8) limits to the AdS metric given in the second line of (A.12). Of course the fact that a singular limit is involved in this procedure means that the AdS and Kaigorodov metrics are inequivalent, as evidenced, for example, by the fact that the AdS metrics have vanishing Weyl tensor while the generalised Kaigorodov metrics do not.

# B Killing vectors and spinors in the Kaigorodov metrics

It is easily seen by inspection that the following are $(\frac{1}{2}n^2 + \frac{3}{2}n + 3)$ Killing vectors of the generalised Kaigorodov metrics:

$$\begin{aligned}
&K_{(0)} = \frac{\partial}{\partial t}\ , \qquad K_{(x)} = \frac{\partial}{\partial x}\ , \qquad K_{(i)} = \frac{\partial}{\partial y^i}\ ,\\
&L_{(i)} = x\,\frac{\partial}{\partial y^i} - y^i\,\frac{\partial}{\partial t}\ , \qquad L_{(ij)} = y^i\,\frac{\partial}{\partial y^j} - y^j\,\frac{\partial}{\partial y^i}\ , &\text{(B.1)}\\
&J = \frac{\partial}{\partial \rho} - a\,t\,\frac{\partial}{\partial t} - b\,x\,\frac{\partial}{\partial x} - c\,y^i\,\frac{\partial}{\partial y^i}\ , &\text{(B.2)}
\end{aligned}$$

where $a$, $b$ and $c$ are given by (A.7). The $K_{(0)}$, $K_{(x)}$ and $K_{(i)}$ Killing vectors mutually commute, and the rest of the algebra of the Killing vectors is

$$\begin{aligned}
&[J, K_{(0)}] = a\,K_{(0)}\ , \qquad [J, K_{(x)}] = b\,K_{(x)}\ , \qquad [J, K_{(i)}] = c\,K_{(i)}\ ,\\
&[J, L_{(i)}] = (a-c)\,L_{(i)}\ , \qquad [J, L_{(ij)}] = 0\ ,\\
&[K_{(x)}, L_{(i)}] = K_{(i)}\ , \qquad [L_{(i)}, K_{(j)}] = \delta_{ij}\,K_{(0)}\ , \qquad [L_{(ij)}, K_{(k)}] = -\delta_{ik}\,K_{(j)} + \delta_{jk}\,K_{(i)}\ ,\\
&[L_{(i)}, L_{(j)}] = 0\ , \qquad [L_{(ij)}, L_{(k)}] = -\delta_{ik}\,L_{(j)} + \delta_{jk}\,L_{(i)}\ ,\\
&[L_{(ij)}, L_{(k\ell)}] = -\delta_{ik}\,L_{(j\ell)} + \delta_{jk}\,L_{(i\ell)} - \delta_{j\ell}\,L_{(ik)} + \delta_{i\ell}\,L_{(jk)}\ . &\text{(B.3)}
\end{aligned}$$

Since the metrics are homogeneous, these symmetries act transitively on the spacetimes. In the four-dimensional case, we have the previously-known five-dimensional group of symmetries on the Kaigorodov spacetime [29].

The Killing spinor equation in a $D$-dimensional spacetime is

$$D_\mu\,\epsilon^\pm = \pm\sqrt{-\frac{\Lambda}{D-1}}\,\Gamma_\mu\,\epsilon^\pm\ , \tag{B.4}$$

where $D_\mu = \partial_\mu + \frac14\omega_\mu^{ab}\,\Gamma_{ab}$. Thus for the generalised Kaigorodov metrics described above, the equation becomes $D_\mu\,\epsilon^\pm = \pm\frac12 c\,\Gamma_\mu\,\epsilon^\pm$.

It is instructive first to consider the integrability conditions for the existence of Killing spinors. The simplest of these is the 2'nd-order condition that results from taking a commutator of the derivatives $\bar D_\mu^\pm \equiv D_\mu \mp \frac12 c\,\Gamma_\mu$ arising in the Killing spinor equation $\bar D_\mu^\pm\,\epsilon^\pm = 0$. Thus we obtain the condition

$$H_{\mu\nu}\,\epsilon^\pm \equiv 4[\bar D_\mu^\pm, \bar D_\nu^\pm]\,\epsilon^\pm = R_{\mu\nu\rho\sigma}\,\Gamma^{\rho\sigma}\,\epsilon^\pm + 2c^2\,\Gamma_{\mu\nu}\,\epsilon^\pm = 0\ . \tag{B.5}$$

Note that the quantity $H_{\mu\nu}$ can be written simply as $H_{\mu\nu} = C_{\mu\nu\rho\sigma}\,\Gamma^{\rho\sigma}$, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor, discussed in the previous section. Upon substitution of the Riemann tensor, (B.5) gives algebraic conditions on the Killing spinors $\epsilon^\pm$, in the form of projection operators formed from the $\Gamma$ matrices. These are necessary conditions for the existence of Killing spinors, and in many cases they are also sufficient. (See [47] for a discussion of higher-order integrability conditions for Killing spinors.) However, as we shall see, for the generalised Kaigorodov metrics these 2'nd-order integrability conditions are not in fact sufficient. Before proceeding to study (B.5), therefore, let us present also the 3'rd-order integrability condition that follows by taking a further derivative of (B.5), and using the original Killing spinor equation again. Thus we obtain

$$H_{\lambda\mu\nu}^\pm\,\epsilon^\pm \equiv (\nabla_\lambda\,R_{\mu\nu\rho\lambda})\Gamma^{\rho\sigma}\,\epsilon^\pm \mp 2c\,R_{\mu\nu\lambda\rho}\,\Gamma^\rho\,\epsilon^\pm \pm 2c^3\,(g_{\nu\lambda}\,\Gamma_\mu - g_{\mu\lambda}\,\Gamma_\nu)\,\epsilon^\pm = 0\ . \tag{B.6}$$

From (A.4), and substituting the solution for $a$, $b$ and $c$ in (A.7), we find that the quantities $H_{\mu\nu}$ in the 2'nd-order integrability condition (B.5) are given by

$$\begin{aligned}
&H_{02} = 2n(n+2)\,L^2\,(\Gamma_{02}+\Gamma_{12})\ , \qquad H_{0i} = -2(n+2)\,L^2\,(\Gamma_{0i}+\Gamma_{1i})\ ,\\
&H_{12} = -2n(n+2)\,L^2\,(\Gamma_{02}+\Gamma_{12})\ , \qquad H_{i1} = 2(n+2)\,L^2\,(\Gamma_{0i}+\Gamma_{1i})\ , \qquad \text{(B.7)}\\
&H_{01} = 0\ , \qquad H_{ij} = 0\ , \qquad H_{2i} = 0\ . \qquad \text{(B.8)}
\end{aligned}$$

Thus the integrability conditions $H_{\mu\nu}\,\epsilon^\pm = 0$ imply that $\epsilon^\pm$ must satisfy

$$\Gamma_{01}\,\epsilon^\pm = \epsilon^\pm\ . \tag{B.9}$$

One might be tempted to think that this were the only condition, in which case the Killing spinors would preserve half of the maximal supersymmetry. However, as foreshadowed above, higher-order integrability can place further constraints in certain cases, and in fact the present example is one such. It suffices to consider just one special case among the 3'rd order conditions implied by (B.6). Consider, for example, the components $H^{\pm}_{10i}$ of $H^{\pm}_{\lambda\mu\nu}$. The covariant derivative of the Riemann tensor can be evaluated using the general expression $\nabla_\mu V_\nu = \partial_\mu V_\nu + (\omega_\nu{}^\rho)_\mu V_\rho$, and after some algebra we find that $\nabla_1 R_{0i\rho\sigma}\Gamma^{\rho\sigma} = 8(n+2)\,L^2\,\Gamma_{2i}$. Substituting into (B.6), we therefore find that $H^{\pm}_{10i} = 8(n+2)\,L^2\,(\Gamma_{2i} \pm \Gamma_i)$, implying that $\epsilon^{\pm}$ must also satisfy the condition

$$\Gamma_2\,\epsilon^{\pm} = \pm\epsilon^{\pm}\ , \tag{B.10}$$

in addition to (B.9). In principle we should examine all the components of $H^{\pm}_{\lambda\mu\nu}$, but the upshot is that no further conditions result. The simplest way to prove this is by moving now to an explicit construction of Killing spinors that satisfy the two conditions (B.9) and (B.10).

Substituting the spin connection (A.3) into this, we find that the Killing spinors must satisfy the following system of equations:

$$\begin{aligned}
\frac{\partial\epsilon^{\pm}}{\partial t} + \tfrac14(a+b)\,e^{a\,\rho}\,(\Gamma_{02}+\Gamma_{12})\,\epsilon^{\pm} &= \pm\tfrac12 c\,e^{a\,\rho}\,(\Gamma_0+\Gamma_1)\,\epsilon^{\pm}\ ,\\
\frac{\partial\epsilon^{\pm}}{\partial x} + e^{b\,\rho}\,\Big(\tfrac12 b\,\Gamma_{12} - \tfrac14(a-b)\,\Gamma_{02}\Big)\,\epsilon^{\pm} &= \pm\tfrac12 c\,e^{b\,\rho}\,\Gamma_1\,\epsilon^{\pm}\ ,\\
\frac{\partial\epsilon^{\pm}}{\partial y^i} - \tfrac12 c\,e^{c\,\rho}\,\Gamma_{2i}\,\epsilon^{\pm} &= \pm\tfrac12 c\,e^{c\,\rho}\,\Gamma_i\,\epsilon^{\pm}\ ,\\
\frac{\partial\epsilon^{\pm}}{\partial\rho} - \tfrac14(a-b)\,\Gamma_{01}\,\epsilon^{\pm} &= \pm\tfrac12 c\,\Gamma_2\,\epsilon^{\pm}\ .
\end{aligned} \tag{B.11}$$

It is easily seen from these equations and from (A.7) that the Killing spinors are given by $\epsilon^{\pm} = e^{\frac12(n+4)L\,\rho}\,\epsilon^{\pm}_0$, where $\epsilon^{\pm}_0$ is any constant spinor that satisfies the conditions

$$\Gamma_2\,\epsilon^{\pm}_0 = \pm\epsilon^{\pm}_0\ ,\qquad \Gamma_{01}\,\epsilon^{\pm}_0 = \epsilon^{\pm}_0\ , \tag{B.12}$$

implying that $\epsilon^{\pm}$ satisfies the conditions (B.9) and (B.10) that followed from integrability. Thus by combining the necessary conditions coming from integrability with the sufficient conditions coming from the explicit solutions, we conclude that we have found the general solutions for the Killing spinors in the generalisation of the Kaigorodov spacetime, and that they preserve 1/4 of the supersymmetry.

## C Generalisations of Carter-Novotný-Horský metrics

The form of the metrics that arise in the spherical reduction of the non-extremal $p$-brane plus wave solutions is

$$ds^2 = c_1\, e^{2\tilde{d}\rho/d} \Big( - K^{-1}\, e^{2f}\, dt^2 + K\, (dx_1 + \coth\mu_2\, (K^{-1} - 1) dt)^2 + dy^i\, dy^i \Big) + c_2\, e^{-2f}\, d\rho^2\ , \tag{C.1}$$

(see, for example, (2.20) or (3.12)), where $e^{2f} = 1 - k\, e^{-\tilde{d}\rho}$ and $K = 1 + k\, \sinh^2\mu_2\, e^{-\tilde{d}\rho}$. (We are setting the gravitational constants $\kappa$ to unity here, for convenience.) Let us consider two cases. First, when the charge for the harmonic function $K$ associated with the wave is zero, so that $K = 1$, we have $\mu_2 = 0$ and hence the metric (C.1) becomes

$$ds^2 = c_1\, e^{2\tilde{d}\rho/d} \Big( - e^{2f}\, dt^2 + dx_1^2 + dy^i\, dy^i \Big) + c_2\, e^{-2f}\, d\rho^2\ . \tag{C.2}$$

Now, consider instead the case where the charge associated with the wave is non-zero. Let us make the following Lorentz boost on the coordinates $(t, x_1)$:

$$\begin{aligned} x_1 &= x_1'\, \cosh\mu_2 + t'\, \sinh\mu_2\ , \\ t' &= x_1'\, \sinh\mu_2 + t'\, \cosh\mu_2\ . \end{aligned} \tag{C.3}$$

Note that in terms of the velocity $v$ for the boost along $x_1$, we have simply

$$x_1 = \gamma\, (x_1' - v\, t')\ , \qquad t = \gamma\, (t' - v\, x_1')\ , \tag{C.4}$$

where

$$v = \tanh\mu_2\ , \qquad \gamma = (1 - v^2)^{-1/2} = \cosh\mu_2\ . \tag{C.5}$$

After simple algebra, we find that the metric (C.1) becomes

$$ds^2 = c_1\, e^{2\tilde{d}\rho/d} \Big( - e^{2f}\, dt'^2 + {dx_1'}^2 + dy^i\, dy^i \Big) + c_2\, e^{-2f}\, d\rho^2\ , \tag{C.6}$$

which is identical in form to the previously-obtained metric (C.2).[7] Note that the Lorentz boost (C.3) is a valid coordinate transformation only if the coordinate $x_1$ is not periodic,

[7] Of course the Lorentz boost (C.3) can equally well be applied not only to the spherically-reduced metrics considered in this appendix, but also to the original non-extremal $p$-branes with superimposed pp-waves. In fact *any* non-extremal solution with a superimposed pp-wave can be mapped by the transformation (C.3) into a solution where the wave momentum vanishes and hence the associated harmonic function $K$ becomes just the identity. A particularly striking example is when a non-extremal black hole supported only by a charge for a Kaluza-Klein vector is oxidised back to the higher dimension. In this case, the coordinate transformation (C.3) maps the higher-dimensional wave metric into a purely Minkowski metric. An example of this is the non-extremal D0-brane in $D = 10$, which, after oxidation to a wave in $D = 11$, can then be mapped into Minkowski spacetime. Note that this cannot be done in the extremal limit, since the boost transformation (C.3) then becomes singular.

but instead ranges over the entire real line. Thus it is only when $x_1$ is non-periodic that the effect of the Kaluza-Klein charge can be "undone" by the Lorentz boost (C.3).

If $x_1$ is non-periodic, the spherical reductions of the non-extremal pure M2-brane, M5-brane and D3-brane are identical, up to a Lorentz boost in the $(t, x_1)$ plane, to the spherical reductions of the non-extremal M2-brane, M5-brane and D3-brane that also have a pp wave propagating on the world volume. The Lorentz boost that relates the two cases becomes an infinite boost in the extremal limit $\mu_2 \to \infty$. This explains why the spherical reductions give two distinct types of metric in the extremal cases, namely AdS if there is no pp-wave, and the generalised Kaigorodov metric if there is a pp-wave propagating on the original $p$-brane world-volume.

In the non-extremal case, we have seen from the above discussion that there is just the one type of metric to consider after spherical reduction, namely (C.2), regardless of whether or not there is a pp-wave propagating on the original $p$-brane.

Thus the general class of $D = n+3$ dimensional metrics that arises by the spherical dimensional reduction of non-extremal boosted $p$-branes is included in the class of metrics

$$ds^2 = -e^{2a\rho+2f}\, dt^2 + e^{2b\rho}\, (dx + e^{(a-b)\rho}\, dt)^2 + e^{2c\rho}\, dy^i\, dy^i + e^{-2f}\, d\rho^2 \ , \tag{C.7}$$

where $a$, $b$ and $c$ are constants, and the function $f$ is given by

$$e^{2f} = 1 - k\, e^{-(a-b)\rho} \ . \tag{C.8}$$

(In obtaining the metric form (C.7) from (C.1), we have performed coordinate transformations that would not be valid globally if $x_1$ were a periodic coordinate. For the present purposes we are principally concerned with local properties of the metrics, for which this point is not essential. If $x_1$ is non-compact, ranging over the entire real line, the transformations are in any case globally valid.)

We find that (C.7) is an dimensional Einstein metric if the constants $a$, $b$ and $c$ are given, as previously in the generalised Kaigorodov metrics in (A.7), by

$$a = (n+4)\, L \ , \qquad b = -n\, L \ , \qquad c = 2L \ , \tag{C.9}$$

where again the cosmological constant $\Lambda$ is related to $L$ by $L = \frac{1}{2}\sqrt{-\Lambda/(n+2)}$. Note that the Einstein metrics can then be written in the form

$$ds^2 = e^{-2nL\rho}\, dx^2 + e^{4L\rho}\, (2\, dx\, dt + k\, dt^2 + dy^i\, dy^i) + (1 - k\, e^{-2(n+2)L\rho})^{-1}\, d\rho^2 \ . \tag{C.10}$$

In the natural orthonormal basis $e^0 = e^{a\rho+f}\, dt$, $e^1 = e^{b\rho}\, (dx + e^{(a-b)\rho}\, dt)$, $e^2 = e^{-f}\, d\rho$, $e^i = e^{c\rho}\, dy^i$, we find that with the constants $a$, $b$ and $c$ taking their Einstein-metric values

(C.9), the curvature 2-forms can be written in terms of the Weyl 2-forms $C_{ab}$ as $\Theta_{ab} = -4L^2\,\eta_{ac}\,\eta_{bd}\,e^c\wedge e^d + C_{ab}$, where $C_{ab} = C_{ab}^{\rm Kiag.} + \widetilde{C}_{ab}$, with $C_{ab}^{\rm Kaig.}$ being the Weyl 2-forms for the Kaigorodov metric, as given by (A.10), and

$$\begin{aligned}
&\widetilde{C}_{01} = 2n\alpha\, e^0\wedge e^1\ , \qquad \widetilde{C}_{02} = 2n\,\alpha\, e^0\wedge e^2 + 2n(n+2)L^2\,(e^f-1)\,e^1\wedge e^2,\\
&\widetilde{C}_{0i} = -4\alpha\, e^0\wedge e^i - 2(n+2)L^2\,(e^f-1)\,e^1\wedge e^i\ ,\\
&\widetilde{C}_{12} = 2n(n+1)\,\alpha\, e^1\wedge e^2 + 2n(n+2)L^2\,(e^f-1)\,e^0\wedge e^2,\\
&\widetilde{C}_{1i} = -2n\,\alpha\, e^1\wedge e^i - 2(n+2)L^2\,(e^f-1)\,e^0\wedge e^i\ ,\\
&\widetilde{C}_{2i} = -2n\alpha\, e^2\wedge e^i\ , \qquad \widetilde{C}_{ij} = 4\alpha\, e^i\wedge e^j\ ,
\end{aligned} \tag{C.11}$$

where $\alpha = k\,L^2\,e^{-2(n+2)\rho}$. Note that with the generalisations of the Carter-Novotný-Horský metrics written in the form (C.10), we regain the generalisations (A.8) of the Kaigorodov metric simply by setting $k=0$. It can be seen from (C.11) that the Weyl tensor reduces to (A.10) in this limit.

With the metric written in the form (C.10), it is easy to write down the Killing vectors. First of all, there are manifest shift symmetries for all the coordinates $x$, $t$ and $y^i$. In addition, there are certain rotational symmetries in the $(t, x, y^i)$ hyperplane. Thus there are in total $(\frac12 n^2 + \frac32 n + 2)$ Killing vectors, given by

$$\begin{aligned}
&K_{(0)} = \frac{\partial}{\partial t}\ , \qquad K_{(x)} = \frac{\partial}{\partial x}\ , \qquad K_{(i)} = \frac{\partial}{\partial y^i}\ ,\\
&L_{(i)} = y^i\,\frac{\partial}{\partial t} - (x+k\,t)\,\frac{\partial}{\partial y^i}\ , \qquad L_{(ij)} = y^i\,\frac{\partial}{\partial y^j} - y^j\,\frac{\partial}{\partial y^i}\ .
\end{aligned} \tag{C.12}$$

Note that these non-extremal metrics have one less Killing vector than the extremal generalised Kaigorodov metrics discussed previously, since there is no longer a symmetry under which the coordinate $\rho$ is shifted while making compensating scale transformations of the other coordinates. Thus there is no longer an analogue of the $J$ Killing vector in (B.1) in this case. In the four-dimensional case, there are now four Killing vectors. Note that in all dimensions the non-extremal Einstein metrics (C.10) are inhomogeneous. This can be seen by calculating the curvature invariant $R^{\mu\nu\rho\sigma}\,R_{\mu\nu\rho\sigma}$, which turns out to be dependent on the coordinate $\rho$.

As one would expect for spacetimes coming from the dimensional reduction of non-extremal solutions, there are no Killing spinors in the generalised Carter-Novotný-Horský metrics.

The four-dimensional case ($n=1$) of the above metrics corresponds to a previously-encountered solution. In general, for arbitrary $n$, let us define a new radial coordinate $R$, related to $\rho$ by

$$e^{(n+2)L\rho} = \sqrt{k}\, \cosh((n+2)LR)\ . \tag{C.13}$$

In terms of $R$, the metrics (C.10) become

$$\begin{aligned} ds^2 &= \Big(\sqrt{k}\, \cosh((n+2)LR)\Big)^{4/(n+2)} \Big[k^{-1}\,(dx + kdt)^2 - k^{-1}\,\tanh^2((n+2)LR)\, dx^2 \\ &\quad + dy^i\, dy^i\Big] + dR^2\ . \end{aligned} \tag{C.14}$$

It is now easily seen that after simple coordinate transformations, this metric in the four-dimensional case $n = 1$ becomes equivalent to the metric given in (13.32) of [31], which was found in this form by Novotný and Horský [36]. It is a special case of a general class of four-dimensional Einstein metrics found by Carter in [35].

In three dimensions, the Carter-Novotný-Horský metric becomes simply $ds^2 = dx^2 + e^{4L\rho}\,(2dx\, dt + k\, dt^2) + (1 - k\, e^{-4L\rho})^{-1}\, d\rho^2$. This can be seen to be equivalent to the BTZ black-hole metric described in [19].

## References


[1] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, Phys. Lett. **B379** (1996) 99, hep-th/9601029.

[2] C.G. Callan and J.M. Maldacena, *D-brane approach to black hole quantum mechanics*, Nucl. Phys. **B472** (1996) 591, hep-th/9602043.

[3] G. Horowitz and A. Strominger, *Counting states of near-extremal black holes*, Phys. Rev. Lett. **77** (1996) 2368, hep-th/9602051.

[4] J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa. *D-branes and spinning black holes*, Phys. Lett. **B391** (1997) 93, hep-th/9602065.

[5] J.C. Breckenridge, D.A. Lowe, R.C. Myers, A.W. Peet, A. Strominger and C. Vafa. *Macroscopic and microscopic entropy of near-extremal spinning black holes*, Phys. Lett. **B381** (1996) 423, hep-th/9603078.

[6] M. Cvetič and A.A . Tseytlin, *Solitonic strings and BPS saturated dyonic black holes*, Phys. Rev. **D53** (1996) 5619, Erratum-ibid. **D55** (1997) 3907, hep-th/951203.

[7] A.A. Tseytlin, *Extreme dyonic black holes in string theory*, Mod. Phys. Lett. **A11** (1996) 689, hep-th/9601177.

[8] S.R. Das, *Black hole entropy and string theory*, hep-th/9602172.

[9] A.A. Tseytlin, *Extremal black hole entropy from string sigma model*, Nucl. Phys. **B477** (1996) 431, hep-th/9605091.

[10] M. Cvetič and A.A. Tseytlin, *Sigma model of near-extreme rotating black holes and their microstates*, hep-th/9806141.

[11] S.S. Gubser, I.R. Klebanov and A.W. Peet, *Entropy and temperature of black 3-branes*, Phys. Rev. **D54** (1996) 3915, hep-th/9602135.

[12] I.R. Klebanov and A.A. Tseytlin, *Entropy of near-extremal black p-branes*, Nucl. Phys. **B475** (1996) 164, hep-th/9604089.

[13] I.R. Klebanov and A.A. Tseytlin, *Near-extremal black hole entropy and fluctuating 3-branes*, Nucl. Phys. **B479** (1996) 319, hep-th/9607107.

[14] J. Maldacena, *The large $N$ limit of superconformal field theories and supergravity*, hep-th/9711200.

[15] I.R. Klebanov, *World volume approach to absorption by non-dilatonic branes*, Nucl. Phys. **B496** 231, hep-th/9702076.

[16] S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *String theory and classical absorption by three-branes*, Nucl. Phys. **B499** (1997) 217, hep-th/9703040.

[17] S. Hyun, *U-duality between three and higher dimensional black holes*, hep-th/9704005.

[18] K. Sfetsos and A. Skenderis. *Microscopic derivation of the Bekenstein-Hawking entropy for non-extremal black holes*, Nucl. Phys. **B517** (1998) 179, hep-th/9711138.

[19] M. Banados, C. Teitelboim and J. Zanelli, *The black hole in three dimensional space time*, Phys. Rev. Lett. **69** (1992) 1849, hep-th/9204099.

[20] G.T. Horowitz and D.L. Welch, *Exact three-dimensional black holes in string theory*, Phys. Rev. Lett. **71** (1993) 328, hep-th/9302126.

[21] A. Strominger, *Black hole entropy from near horizon microstates*, JHEP **02** (1998) 009, hep-th/9712251.

[22] D. Birmingham, I. Sachs and S. Sen, *Entropy of three-dimensional black holes in string theory*, hep-th/9801019.

[23] M. Cvetič and F. Larsen, *Near horizon geometry of rotating black holes in five dimensions*, hep-th/9805097.

[24] V. Balasubramanian and F. Larsen, *Near horizon geometry and black holes in four dimensions*, hep-th/9802198.

[25] M. Cvetič and F. Larsen. *Microstates and near-horizon geometry for rotating black holes in four dimensions*, hep-th/9805146.

[26] J.D. Brown and M. Henneaux, *Central charges in the canonical realization of asymptotic symmetries: an example from three-dimensional gravity*, Comm. Math. Phys. **104** (1986) 207.

[27] M.J. Duff, G.W. Gibbons and P.K. Townsend, *Macroscopic superstrings as interpolating solitons*, Phys. Lett. **B332** (1994) 321, hep-th/9405124.

[28] G.W. Gibbons, G.T. Horowitz and P.K. Townsend, *Higher-dimensional resolution of dilatonic black hole singularities*, Class. Quant. Grav. **12** (1995) 297, hep-th/9410073.

[29] V.R. Kaigorodov, *Einstein spaces of maximum mobility*, Dokl. Akad. Nauk. SSSR **146** (1962) 793; Sov. Phys. Doklady **7** (1963) 893.

[30] S.T.C. Siklos, in *Galaxies, axisymmetric systems and relativity*, ed. M.A.H. MacCallum (Cambridge University Press, 1985).

[31] D. Kramer, H. Stephani, E. Herlt and M.A.H. MacCallum, *Exact solutions of Einstein's field equations*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1980).

[32] J. Podolský, *Interpretation of the Siklos solutions as exact gravitational waves in the anti-de Sitter universe*, gr-qc/9801052.

[33] G.T. Horowitz and E.J. Martinec, *Comments on black holes in matrix theory*, Phys. Rev. **D57** (1998) 4935, hep-th/9710217.

[34] S.R. Das, S.D. Mathur, S.K. Rama and P. Ramadevi, *Boots, Schwarzschild black holes and absorbtion cross-sections in M theory*, Nucl. Phys. **B527** (1998) 187, hep-th/9711003.

[35] B. Carter, *A new family of Einstein spaces*, Phys. Lett. **A26** (1968) 399.

[36] J. Novotný and J. Horský, *On the plane gravitational condensor with the positive gravitational constant*, Czech. J. Phys. **B24** (1974) 718.

[37] H. Lü and C.N. Pope, *p-brane solitons in maximal supergravities*, Nucl. Phys. **B465** (1996) 127, hep-th/9512012.

[38] M. Cvetič and C.M. Hull, *Black holes and U duality*, Nucl. Phys. **B480** (1996) 296, hep-th/9606193.

[39] H. Lü, C.N. Pope and K.S. Stelle, *Weyl group invariance and p-brane multiplets*, Nucl. Phys. **B476** (1996) 89, hep-th/9602140.

[40] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities*, Nucl. Phys. **B523** (1998) 73, hep-th/9710119.

[41] H.J. Boonstra, B. Peeters, K. Skenderis, *Duality and asymptotic geometries*, Phys. Lett. **B411** (1997) 59, hep-th/9706192.

[42] E. Bergshoeff and K. Behrndt, *D-instantons and asymp- totic geometries*, Class. Quant. Grav. **15** (1998) 1801, hep-th/9803090.

[43] E. Cremmer, I.V. Lavrinenko, H. Lu, C.N. Pope, K.S. Stelle and T.A. Tran, *Euclidean signature supergravities, dualities and instantons*, hep-th/9803259, to appear in Nucl. Phys. **B**.

[44] M. Bremer, M.J. Duff, H. Lü, C.N. Pope and K.S. Stelle, *Instanton cosmology and domain walls from M-theory and string theory*, hep-th/9807051.

[45] M.J. Duff, H. Lü and C.N. Pope, *The black branes of M-theory*, Phys. Lett. **B382** (1996) 73, hep-th/9604052.

[46] M. Cvetič and A.A. Tseytlin, *Non-extreme black holes from non-extreme intersecting M-branes*, Nucl. Phys. **B478** (1996) 181, hep-th/9606033.

[47] N.P. Warner and P. van Nieuwenhuizen, *Integrability conditions for Killing spinors*, Comm. Math. Phys. **93** (1984) 277.